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Kinetic theory of 2D point vortices from a BBGKY-like hierarchy

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Abstract

Starting from the Liouville equation, we derive the exact hierarchy of equations satisfied by the reduced distribution functions of the single species point vortex gas in two dimensions. Considering an expansion of the solutions in powers of $1/N$ (where N is the number of vortices) in a proper thermodynamic limit $N \to +\infty$, and neglecting some collective effects, we derive a kinetic equation satisfied by the smooth vorticity field which is valid at order $O(1/N)$. This equation was obtained previously [P.H. Chavanis, Phys. Rev. E 64 (2001) 026309] from a more abstract projection operator formalism. If we consider axisymmetric flows and make a Markovian approximation, we obtain a simpler kinetic equation which can be studied in great detail. We discuss the properties of these kinetic equations in regard to the H-theorem and the convergence (or not) towards the statistical equilibrium state. We also study the growth of correlations by explicitly calculating the time evolution of the two-body correlation function in the linear regime. In a second part of the paper, we consider the relaxation of a test vortex in a bath of field vortices and obtain the Fokker–Planck equation by directly calculating the second (diffusion) and first (drift) moments of the increment of position of the test vortex. A specificity of our approach is to obtain general equations, with a clear physical meaning, that are valid for flows that are not necessarily axisymmetric and that take into account non-Markovian effects. A limitation of our approach, however, is that it ignores collective effects.

Keywords: Vortex dynamics; Long-range interactions; Statistical mechanics; Kinetic theory

1. Introduction

Several authors have wondered whether fluid turbulence could be described in terms of statistical mechanics [1]. 3D turbulence has been attacked by different methods [2–5] inspired by statistical mechanics and kinetic theories. Some progress has also been made in the simpler case of 2D turbulence (see the reviews in Refs. [6–8]). 2D turbulence is not just academic but is relevant to describe geophysical and astrophysical flows. 2D flows are characterized by the spontaneous formation of large-scale vortices that dominate the dynamics [9,10]. The most famous example is Jupiter's great red spot, a huge vortex persisting for more than three centuries in a turbulent shear layer between two zonal jets in the southern hemisphere of the planet [11]. Other examples of this self-organization are the cyclones and anticyclones in the earth atmosphere, the jets in the oceans like the gulf stream or the intense jets on Jupiter [12]. As a first step to tackle the problem, it can be of interest to study the dynamics of a system of N point vortices on

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a plane [13]. Each vortex produces a velocity field that moves the other vortices in a self-consistent manner. The velocity created by a vortex decreases like $1/r$ which is similar to the Coulombian or Newtonian interaction in two dimensions. Therefore, the interaction between point vortices is long-range, like the interaction between stars in a galaxy or between electric charges in a plasma. Note, however, that point vortices produce a velocity while material particles produce a force (acceleration). Apart from this (important) difference, the point vortex gas has a Hamiltonian structure [14] and we can try to apply the methods of statistical mechanics and kinetic theory to that system. Therefore, the N-vortex problem [13] is of fundamental interest in statistical mechanics and kinetic theory. It provides a physical example of systems with long-range interactions, whose dynamics and thermodynamics are actively studied at present [15].

The statistical mechanics of 2D point vortices was first considered by Onsager [16] in a seminal paper. He showed that statistical equilibrium states with sufficiently large energies have negative temperatures. For such states, like-sign vortices have the tendency to group themselves and form clusters. If the circulations of all the point vortices have the same sign, the equilibrium state is a large-scale vortex (supervortex) similar to vortices observed in geophysical and astrophysical flows. When the point vortices have positive and negative circulations, the equilibrium state is generically a dipole made of a cluster of positive vortices and a cluster of negative vortices. The pioneering work of Onsager was pursued by Joyce and Montgomery [17] and Lundgren and Pointin [18], using a mean field approximation. Using a combinatorial analysis, Joyce and Montgomery introduced an entropy for the point vortex gas which is similar to the Boltzmann entropy for material particles. The statistical equilibrium state (most probable) is obtained by maximizing this Boltzmann entropy while conserving all the constraints imposed by the dynamics (total number N of point vortices and energy E, as well as angular momentum L and impulse $\mathbf{P}$ for domains with a special symmetry). For point vortices with equal circulation γ, the smooth vorticity field is given by the Boltzmann distribution $\omega(\mathbf{r}) = A e^{-\beta\gamma\psi(\mathbf{r})}$, where the potential is played by the stream function $\psi(\mathbf{r})$. Using $\omega = -\Delta\psi$, the stream function is then determined by the Boltzmann–Poisson equation. Lundgren and Pointin started from the exact equilibrium hierarchy of equations satisfied by the reduced distribution functions $P_j(\mathbf{r}_1, \ldots, \mathbf{r}_j)$ of the point vortex gas and, by neglecting all the correlations between point vortices, derived a differential equation determining the equilibrium distribution of the one-body distribution function $P_1(\mathbf{r}_1)$. Using the fact that $\omega(\mathbf{r}) = N\gamma P_1(\mathbf{r})$, the mean field equation derived by Lundgren and Pointin coincides with the Boltzmann–Poisson equation derived by Joyce and Montgomery. In a mathematical work, Caglioti et al. [19] showed rigorously that the mean field approximation is exact in a proper thermodynamic limit $N \to +\infty$ such that $\gamma \sim 1/N$, $E \sim 1$, $\beta \sim N$ and $V \sim 1$ (where V is the area of the domain). In that limit the N-body distribution at statistical equilibrium is a product $P_N(\mathbf{r}_1, \ldots \mathbf{r}_N) = P_1(\mathbf{r}_1) \ldots P_1(\mathbf{r}_N)$ of N one-body distributions that are solution of the Boltzmann–Poisson equation. This statistical equilibrium state is expected to be achieved for $t \to +\infty$. We stress, however, that the statistical theory is based on the assumption that "at statistical equilibrium, all accessible microstates are equiprobable". This is essentially a postulate, so there is no guarantee that the point vortex gas will reach a statistical equilibrium state of the form described above (based on the microcanonical distribution). In order to determine the timescale of the relaxation of the smooth vorticity field $\omega(\mathbf{r}, t)$, and in order to establish whether (or not) the system will truly relax towards Boltzmann statistical equilibrium, we must develop a kinetic theory of point vortices.

A kinetic theory was developed by Dubin and O'Neil [20] in the case of a non-neutral plasma confined by a strong magnetic field, a system isomorphic to the point vortex gas. They started from the Klimontovich equation and used a quasilinear approximation to determine the current of the smooth density due to discrete interactions between point vortices. They considered an axisymmetric evolution of the system and, in the course of their derivation, made a Markov approximation assuming that the two-body correlation function relaxes on a timescale that is much shorter than the timescale on which the smooth density field changes (this is the counterpart of the Bogoliubov hypothesis in plasma physics). They obtained a closed expression of the current, see Eq. 11 of Ref. [20], taking into account "collective effects" between the particles. These collective effects are similar to those giving rise to the Debye shielding in plasma physics in the Lenard–Balescu approach [21,22]. In plasma physics, they take into account the fact that a charge is surrounded by a polarization cloud of opposite charges. In the case of point vortices, their physical interpretation and their consequence is more difficult to establish.

A kinetic theory of point vortices was carried out independently by Chavanis [23], using an analogy with the kinetic theory developed for stellar systems. He started from the Liouville equation and used the projection operator formalism of Willis and Picard [24] to derive a kinetic equation for the smooth vorticity distribution $\omega(\mathbf{r}, t)$. By this