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Physica A 387 (2008) 1404–1410

PHYSICA A

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Susceptibility of the Ising model on the scale-free network with a Cayley tree-like structure

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Received 8 June 2007; received in revised form 15 October 2007
Available online 22 October 2007

Abstract

We derive the exact expression for the zero-field susceptibility of each spin of the Ising model on the scale-free (SF) network having the degree distribution $P(k) \propto k^{-\gamma}$ with the Cayley tree-like structure. The system shows that: (i) the zero-field susceptibility of a spin in the interior part diverges below the transition temperature of the SF network with the Bethe lattice-like structure T_c for $\gamma > 3$, while it diverges at any finite temperature for $\gamma \leq 3$, and (ii) the surface part diverges below the divergence temperature of the SF network with the Cayley tree-like structure T_s for $\gamma > 3$, while it diverges at any finite temperature for $\gamma \leq 3$.

PACS: 05.40.-a; 05.50.+q; 89.75.Hc

Keywords: Networks; Scale-free networks; Cayley tree; Ising model; Magnetic susceptibility

1. Introduction

In recent years, scale-free networks (SFNs) have been studied as models to describe topologically complex real-world systems, *e.g.*, social systems, protein-interaction networks, the internet, and the worldwide web [1,2]. An SFN is characterized by a network having a power-law degree distribution $P(k) \propto k^{-\gamma}$, where degree k is the number of edges connected to a vertex, and the network topology affects various processes taking place there, *e.g.*, network failure, spread of infections, and cooperative behaviors of the interacting systems [3–17]. Among them, the Ising model, a basic model of the interacting systems, on the SFNs has been much investigated by both analytical [3–10] and numerical methods [11–15] to show that their critical behaviors are far from those on periodic lattices and strongly depend on the exponent γ.

In analyzing networks, it has been frequently assumed that the cyclic paths can be ignored. Then we can classify a network with no cyclic paths into the following two tree-like structures [18]: (i) the Bethe lattice-like structure (BLS) where the depth is infinite and there is no boundary, and (ii) the Cayley tree-like structure (CTS) where the depth is finite and the tree is bounded by leaves. Recently, the present authors showed that the ferromagnetic Ising model on the SFN with the CTS shows behaviors entirely different from those with the BLS [10]: (i) The Ising model on the

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2007.10.041

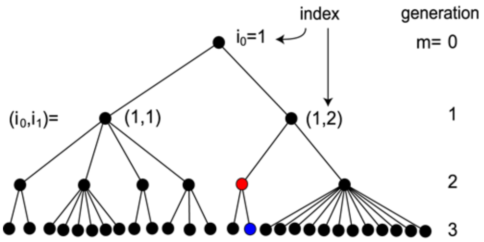

Fig. 1. An example of an SF Cayley tree with the radius $R = 3$. The index of the red vertex is (1, 2, 1), and that of the blue vertex is (1, 2, 1, 2).(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

SFN with the BLS remains in the ferromagnetic phase at any finite temperature for $\gamma \le 3$, while the phase transition exists at a finite temperature T_c for $\gamma > 3$ and its critical exponents vary depending on the exponent γ [6]. The transition temperature T_c is given by $\tanh(J/T_c) = \langle k\rangle/\langle k(k-1)\rangle$, where J (>0) is the ferromagnetic interaction and $\langle\cdots\rangle$ denotes the average with $P(k)$. (ii) The Ising model on the SFN with the CTS has no magnetization at any finite temperature, and the system susceptibility diverges below a certain temperature (the divergence temperature) T_s depending on the exponent γ. The divergence temperature T_s is given by $\tanh^2(J/T_s) = \langle k\rangle/\langle k(k-1)\rangle$ for $\gamma > 4$, while T_s reaches the infinity for $\gamma \le 4$ [10]. The simplicity of the SFN models with the BLS enables us to perform detailed analysis for local properties of spins deep within the network, while the spin systems on the SFN with the CTS shed lights on the influence of leaves, which are abundant in real networks, on the system's behaviors.

In this paper, we proceed some analytic calculations about the zero-field susceptibility of the Ising model on the SFN with CTS. We derive the exact representations for the one-spin susceptibility on the Cayley tree with any branching structure. Although Matsuda derived the exact representations for the one-spin susceptibility on the regular Cayley tree [19], we use an alternative approach to obtain more generalized form. We show that for the Ising model on the SFN with the CTS: (i) the zero-field susceptibility of a spin in the interior part diverges below the transition temperature of the SFN with the BLS T_c for $\gamma > 3$, while it diverges at any finite temperature for $\gamma \le 3$, and (ii) the zero-field susceptibility in the surface part diverges below the divergence temperature of the SFN with the CTS T_s for $\gamma > 3$, while it diverges at any finite temperature for $\gamma \le 3$. These results mean that the one-spin susceptibility diverges everywhere on the network at any finite temperature for $\gamma \le 3$ but not for $3 < \gamma \le 4$.

2. Scale-free Cayley tree

We can produce *a SF Cayley tree with the radius R* from the configuration model (Fig. 1) [10]. A SF Cayley tree is a random branching tree with the following properties; (i) the mean degree of the vertex on the generation 0 (root) over graph realizations is $\langle k\rangle$, (ii) the mean degree of vertices on the generation n $(1 \le n \le R-1)$ is $\langle k(k-1)\rangle/\langle k\rangle$, and (iii) the degree of vertices on the generation R (leaf) is 1. It reduces to the model analyzed by Dorogovtsev et al., which we call *a SF Bethe lattice*, if we take the radius R the infinity and ignore the leaf-spins on the boundary. In this paper we append the indices to the vertices as follows: We denote: (i) the root by $\vec{i}_0 = (i_0 = 1)$, (ii) the vertices on the generation 1 by $\vec{i}_1 = (\vec{i}_0, i_1) = (i_0, i_1)$ where the number of index i_1 is equal to the number of branches emerging from the root $\vec{i}_0$, (iii) the vertices on the generation 2 by $\vec{i}_2 = (\vec{i}_1, i_2) = (i_0, i_1, i_2)$ where the number of index i_2 with given $\vec{i}_1$ is equal to the number of branches emerging from the vertex $\vec{i}_1$, and so on. The index of each vertex on the nth shell $(1 \le n \le R)$ is given by $\vec{i}_n = (i_0, i_1, i_2, \ldots, i_n)$. We consider the ferromagnetic Ising spin system on an SF Cayley tree with the radius R. The Hamiltonian is

$$H = -J\sum_{\langle ij\rangle} S_i S_j - h\sum_i S_i, \tag{1}$$

where J (>0) is the ferromagnetic interaction, h is the external magnetic field, and S_i (=±1) is the Ising spin variable on the vertex i. The first sum is over all edges of the graph, and the second one is over all vertices.

3. The derivation of the zero-field susceptibility

Matsuda derived the exact expression for the zero-field susceptibility of a spin on any generation of the Ising model on the regular Cayley tree [19]. Here we derive the susceptibility of each spin on the Cayley tree with random