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Mathematical expressions for epidemics and immunization in small-world networks

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Abstract

In this paper we propose an analytic model for describing the epidemic spreading action on the one-dimensional small-world network. Based on the model, the epidemic spreading behaviors without immune strategy and with immune strategy are studied under different conditions. The obtained results suggest that the network's structure is the key factor to influence the time evolution law of the number of all infected vertices when without immune strategy. The infecting percentage with immune strategy has different maximum for different immune probability and triggering time (at which the control strategy is triggered), but for different values of them the infecting percentage reaches each maximum almost at same time. Our analytic model is more convenient and efficient than the previous simulation method.

Keywords: Epidemic spreading; Immunization; Small-world network

1. Introduction

It has long been recognized that the structure of social networks plays an important role in the dynamics of epidemic propagation. These networks exhibit what has been known as the "small-world effect" [1,2], that is any two people can establish contact by going through only a short chain of intermediate acquaintances. Milgram [1] was one of the first to point out the existence of the small-world effect in real population, he performed that the average number of such intermediates is about six—there are "six degrees of separation" between two randomly chosen people in the world. Because of the properties which is "large clustering, small average distance", epidemic can spread very quickly on the small-world network [3]. For instance, SARS [4], foot-and-mouth disease [5], and kinds of plant virtues [6–8], etc., have ever been spreading throughout the whole world quickly, leading to the serious consequence. So looking forward, an efficient control strategy becomes very necessitating.

The small-world model has been introduced by Watts and Strogatz as a simple model of the social network, "W–S model" [3] was defined as follows: they took a one-dimensional regular lattice of N vertices with connections and periodic boundary conditions (the lattices is a ring), then for each of the bonds with some probability "rewire" it. Rewiring means shifting one end of the bond to a new vertex chose uniformly at random from the whole lattice,

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with the exception that no two vertices can have more than one bonds running between them and no vertices can be connected by a bond to itself. Later, Newman and Watts offered another mechanism to generate the small-world model, on the basis of the regular one-dimensional lattice, instead of rewiring each bonds, they added a small number of shortcuts between the chosen pairs of sites randomly, with percentage ϕ per bond on the original regular lattice, which is "N–W model" [9]. For studying the epidemic propagation on the one-dimensional small-world networks with N vertices, L edges are placed on the regular one-dimensional lattice with nearest- and next-nearest-neighbors connections out to some constant range k, so $L = kN$, thus there are ϕL shortcuts in the graph on an average. Newman and Watts have suggested a way to define the average number of neighbors of a given vertex within a neighborhood of r(r is the number of edges which epidemic passed), as follows: $V(r) = \sum_{r'=0}^{r} a(r')[1 + 2\xi^{-1} V(r - r')]$ [9], while $a(r)$ is the number of sites on the underlying lattice at the radius r, ξ denotes the average distance between the ends of shortcuts on the lattices, $\xi = \frac{1}{\phi k}$.

For controlling of infectious epidemic spreading on networks, it is viable to set about from two aspects: one way is to change the structure of networks, that is mainly setting off some vertices or edges, which is called "networks attack"; The other is, toward the model of epidemic propagation on networks, to immune some infectious vertices. So far, there are several immunity strategies having been applied. Recently, a combination model of the two ways is studying for the scale-free network. For the small-world network, Pandit and Amritkar ("P–A model") [10] have offered a control strategy toward the "shortcuts" for the epidemic spreading. They defined the "faraway edges" to identify the "shortcuts" by using "the order of edges", then immune these "shortcuts" randomly. Their studying results showed that it was very useful to immune these shortcuts for the epidemic spreading on small-world networks. After, Motter [11] proposed another way—"the betweenness of edges" [12,13] to identify the "shortcuts" to research the control action of the small-world, his method was also toward the "shortcuts" but for the network itself. All their studies show that the small-world network is very sensitive to these ways toward the "shortcuts". Besides, many researches have studied about the cases that epidemic spreading on small-world networks and scale-free networks [14–19].

Previous works were all realized by simulation on computers, there are few people to use the analytic solution. In this paper, we suggest an analytic solution for the epidemic propagation on one-dimensional small-world networks as well as control action. The basic ingredients of small-world networks are the "shortcuts" connections. So if you want to find an efficient way to control the epidemic spreading, you must make a good method to identify these "shortcuts" at first. In previous works, they all have suggested a method to identify these "shortcuts" firstly, and then, offered the control strategy, but it is not a mediocre problem to identify the "shortcuts". However, our analytic solution could avoid the problem, this analytic function has a simple form only with some parameters, according to it we can get accurate number of infected vertices at every time unit expediently and fleetly. We can also predict the epidemic propagation trend and the epidemic propagation law in different conditions by changing those parameters' value, analytic the reason and key factor of epidemic spreading, it is also conveniently to find the most efficient strategy for controlling and predicting.

2. Describing the spreading dynamics with mathematics

On the one-dimensional "N–W model" small-world network, we study the SI (without immune strategy) and SIR (with immune strategy) model of epidemic spreading. In the network the sites represent the colony's individuals and the bonds represent the contacts between individuals, the sum sites represent the colony size. For the characteristic of the "N–W model" small-world network, the sites on it have two neighborhood relationships: one is the nearest neighbors basing on the regular lattice; we call this "short-edge" neighbor; the other is the neighbors joined by the shortcuts, so call it "far-edge" neighbor. Whereas the structure characters of the small-world network, in order to describe these conveniently, we introduce two nomenclatures "far-edge infecting" and "short-edge infecting". The "short-edge infecting" means the infected vertices which come from sites on the underlying lattice, while "far-edge infecting" means the ones which can be infected via shortcuts, as showing in Fig. 1. So the number of infected vertices caused by epidemic is made up of the two parts. The following part gives the details of epidemic spreading process without immune strategy and with immune strategy.

2.1. The spreading action without immune strategy

Without immune strategy, the individual state are being fit (S) and infected (I), that is SI model. We simply assume that the infecting probability is p $(0 < p < 1)$, the infecting vertex is only affecting the nearest neighbors of it at