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The dynamical temperature and the standard map

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Abstract

Numerical experiments with the standard map at high values of the stochasticity parameter reveal the existence of simple analytical relations connecting the volume and the dynamical temperature of the chaotic component of the phase space.

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1. Introduction

The standard map is an important object in studies in nonlinear dynamics, mainly because it is often used to describe the local behavior of other more complicated symplectic maps [1–3]. What is more, it serves as an important independent mechanical and physical paradigm [4,5]. General properties of the standard map, often in interrelation with those of the separatrix map, Fermi map and other fundamental maps, were considered and studied in detail in Refs. [1–13] and many other works.

The standard map is given by the equations

$$y_{i+1} = y_i + \frac{K}{2\pi}\sin(2\pi x_i)\,(\mathrm{mod}\,1),$$

$$x_{i+1} = x_i + y_{i+1}\,(\mathrm{mod}\,1), \tag{1}$$

where K is the so-called stochasticity parameter [1,2].

In Ref. [10] we studied three major characteristics of the chaotic dynamics of the standard map, namely, the measure μ of the main connected chaotic domain (MCCD), the maximum Lyapunov exponent L of the motion in this domain and the dynamical entropy $h = \mu L$, as functions of the stochasticity parameter K. The perturbations of the domain due to the birth and disintegration of islands of stability, upon variations of K, were considered in particular. By means of extensive numerical experiments, we showed that these perturbations are isentropic, at least approximately: the dynamical entropy follows the smooth dependence

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$h(K) = \ln K/2 + 1/K^2$, and does not fluctuate about it with changing the stochasticity parameter, while local jumps in μ and L due to the birth and disintegration of regular islands are significant.

On the other hand, recently Baldovin [7] and Baldovin et al. [8] introduced the notion of the "dynamical temperature" for the standard map, defining it as the variance of the momentum:

$$T \equiv \langle (y - \langle y \rangle)^2 \rangle - \tfrac{1}{12} = \langle y^2 \rangle - \langle y \rangle^2 - \tfrac{1}{12}, \tag{2}$$

where the angle brackets signify ensemble average; the constant addend is the normalizing shift, introduced for convenience. This choice of normalization gives $T = 0$ for the case of the uniform ensemble [7,8]. Henceforth we use different normalization:

$$T \equiv 12\langle (y - \langle y \rangle)^2 \rangle = 12(\langle y^2 \rangle - \langle y \rangle^2). \tag{3}$$

This gives $T = 1$ for the case of the uniform ensemble. This normalization is used in calculating T in what follows. In practice, the measurement of T is performed by averaging over an iterated trajectory in the chaotic domain, i.e., we assume that the ensemble average is equal to the time average. This is justified in the considered situation, when the regular islands are small and the motion is almost completely ergodic on the phase plane. The relation between ensemble and time averages for low-dimensional symplectic maps in the opposite case of extended borders between chaotic and regular components of the phase space was considered and analyzed in Ref. [7].

In this paper, we investigate the behavior of two state variables of the MCCD of the phase space of the standard map: namely, the volume μ and the dynamical temperature T, both as functions of the stochasticity parameter K. At moderate values of K (at approximately $K < 4$), the non-monotonic (spike) variations in $\mu(K)$ and $T(K)$ are conditioned by the process of absorption of minor chaotic domains by the MCCD, while K increases; at larger values of K (at approximately $K > 4$), they are conditioned by the process of birth and disintegration of stability islands. In other words, the perturbations of the chaotic domain at high values of K are due to the birth and disintegration of islands of stability, upon small variations of K. By means of extensive numerical experiments, we show that the variations of μ and T due to this process obey simple analytical relations.

2. Numerical experiments

The traditional "one trajectory method" (OTM) [1,2,10,14] has been used for calculation of μ. This method consists in computing the number of cells explored by a single trajectory on a grid exposed on the phase plane. A strict but computationally much more expensive approach for measuring μ consists in calculating the values of the coarse-grained area of the chaotic component for a set of various resolutions of the grid, in order to find the asymptotic value of μ at the infinitely fine resolution (see Ref. [12]). In some cases the "Lyapunov characteristic exponent segregation method" (LCESM) [1,2,10,14] has been employed for verification of the obtained values of μ. In this method, a numeric criterion is used for separation of the regular and chaotic trajectories. The criterion is provided by an analysis of the differential distribution of the values of the finite-time Lyapunov characteristic exponents (LCEs) computed on a set of trajectories with the starting values randomly generated on the phase plane or specified on a regular grid on the phase plane. In the distribution, the peak corresponding to the regular trajectories is movable: if one increases the LCE computation time, the peak moves to the left on the abscissa axis, because the computed LCEs for the regular trajectories tend to zero, while all the peaks corresponding to the chaotic domains stay immovable, on condition that the computation time is long enough. By increasing the computation time one can make the movable and immovable peaks completely distinct, and in this way determine the "finite-time LCE" value separating the regular and chaotic trajectories for the given computation time. The OTM and LCESM were both proposed and used by Chirikov [1,2] in computations of μ for the standard map. Analogous methods were used in Ref. [14] in computations of the chaotic domain measure in the Hénon–Heiles problem, and in Ref. [10] for computations of μ for the standard map. A detailed description of the currently used versions of the methods is given in Ref. [10].

In Figs. 1–4, we present the obtained numerical data on the measure μ and the dynamical temperature T of the MCCD, and their interrelations. Fig. 1 shows $\mu(K)$ and $T(K)$ for a perturbation of the MCCD due to the