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Incomplete normalization of probability on multifractals

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Abstract

This work is an extension of the incomplete probability theory from the simple case of monofractals previously studied to the more general case of multifractals that can occur in the phase space without equiprobable partition.

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1. Introduction

The incomplete probability distribution (IPD) has been proposed for the case where the probability cannot be normalized, i.e., $\sum_{i=1}^{w} P_i \neq 1$. For a class of IPD, we have suggested the following incomplete normalization (IN) [1]:

$$\sum_{i=1}^{w} P_i^q = 1. \quad (1)$$

The reader can find discussion of the various reasons for the existence of this nonadditive or unnormalizable probability distribution in Refs. [1,3–7].

One of the cases for which IN is suggested is the chaotic systems evolving in phase space having fractal attractors [2–4]. In our previous papers [3,4], IN was studied on monofractals having homogeneous distribution of segments of same length such as the standard Cantor set in Fig. 1. This set is constructed by iteration. At the kth iteration, the initial line of length L_0 is transformed into a set with $N_k = 2^k$ segments of length $\delta_{ki} = \delta_k = (1/3)^k L_0$ ($i = 1, 2, \ldots, N_k$ are in the order say from left to the right). To understand the link between this structure evolution and the probability distribution of nonequilibrium systems, we can consider an ensemble of chaotic systems which evolve in their phase space [8] and are gradually attracted in a fractal structure (strange attractor) formed by their trajectories. At the same time, the occupied phase volume

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Fig. 1. Standard Cantor set ($m = 2$, $\rho = 1/3$) built at iteration orders from $k = 0$ to $k = 4$. At first iteration, the total length of the curve is $L_1 = (2/3)L_0$ with $N_1 = 2$ segments of length $\delta_1 = (1/3)L_0$; at second iteration, $L_2 = (2/3)^2 L_0$ with $N_2 = 2^2$ segments of length $\delta_2 = (1/3)^2 L_0$, etc.

gradually maps into these attractors. To illustrate this evolution, let us take the example of a phase attractor in the form of the Cantor set in Fig. 1. Suppose L_0 is the volume of the initial states, the total occupied volume at a later time t_k of the kth iteration is $L_k = N_k \delta_k = (2/3)^k L_0$. If all the state points are equally probable, the usual definition of probability that a segment of length δ_{ki} is visited by the system is given by $P_{ki} = \delta_{ki}/L_k$. This probability can be normalized as follows:

$$\sum_{i=1}^{N_k} P_{ki} = \frac{\sum_{i=1}^{N_k} \delta_{ki}}{L_k} = \frac{N_k \delta_k}{L_k} = 1. \tag{2}$$

The probability defined in Eq. (2) is scientifically sound and in agreement with the probability theory. Its only drawback is that it is not convenient for describing physical systems out of equilibrium. If the system is in equilibrium at a stage k, the above definition of probability is reasonable and sufficient for the statistical description of the equilibrium state on the total phase volume L_k. However, if the system is out of equilibrium and L_k is only an intermediate phase volume changing in time, it will be more convenient, in order to take into account the time evolution of the distribution, to define the probability finding a system in δ_{ki} with respect to the initial volume L_0 [1,3], i.e.,

$$p_{ki} = \frac{\delta_{ki}}{L_0}. \tag{3}$$

This probability not only gives us the nonequilibrium distribution at any moment but also the evolution of the distribution with respect to the initial conditions. Notice that this probability is not normalizable since we have $\sum_{i=1}^{N_k} p_{ki} = \left(\sum_{i=1}^{N_k} \delta_{ki}\right) \Big/ (L_0) = (L_k)/(L_0) \neq 1$.

In our previous work [3], it was shown that, for the case of self-similar monofractal, the probability of Eq. (3) could be normalized by

$$\sum_{i=1}^{N_k} p_{ki}^q = 1, \tag{4}$$

for any stage k with a unique $q = d$, where d is the self-similarity dimension of the monofractals (see definition below). The parameter q characterizes the evolution of the phase space. When $q < 1$, the phase volume shrinks in time, when $q > 1$, there is phase space expansion. The normalization becomes complete for $q = 1$ without evolution of distribution. Eq. (4) has been derived on monofractals having segments of same length such as in Fig. 1. It was only conjectured [3] that the same calculation would hold for multifractals which in general contain segments of different lengths, but no proof has been given.

In this work, it will be shown that Eq. (4) holds for any multifractal with self-similar structure having the same iteration rule at each stage. In order to be clear for the readers who have not followed the previous work, we will begin by the simplest case of homogeneous monofractals. More complicated cases with different segments will be analyzed first with two segments of different lengths replacing a segment of precedent stage, and then with an arbitrary number m of different segments. Proof for the uniqueness of q will also be given.