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Stochastic volatility of financial markets as the fluctuating rate of trading: An empirical study

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Abstract

We present an empirical study of the subordination hypothesis for a stochastic time series of a stock price. The fluctuating rate of trading is identified with the stochastic variance of the stock price, as in the continuous-time random walk (CTRW) framework. The probability distribution of the stock price changes (log-returns) for a given number of trades N is found to be approximately Gaussian. The probability distribution of N for a given time interval Δt is non-Poissonian and has an exponential tail for large N and a sharp cutoff for small N. Combining these two distributions produces a non-trivial distribution of log-returns for a given time interval Δt, which has exponential tails and a Gaussian central part, in agreement with empirical observations.

Keywords: Stock market; Stochastic volatility; Subordination; Heston model

1. Stochastic volatility, subordination, and fluctuations in the number of trades

The stock price S_t is a stochastic series in time t. It is commonly characterized by the probability distribution $P_{\Delta t}(x)$ of detrended log-returns $x = \ln(S_{t_2}/S_{t_1}) - \mu\Delta t$, where the time interval $\Delta t = t_2 - t_1$ is called the time lag or time horizon, and μ is the average growth rate. For a simple multiplicative (geometric) random walk, the probability distribution is Gaussian: $P_{\Delta t}(x) \propto \exp(-x^2/2v\Delta t)$, where $v = \sigma^2$ is the variance, and σ is the volatility. However, the empirically observed probability distribution of log-returns is not Gaussian. It is well known that the distribution has power-law tails for large x [1,2]. However, the distribution is also non-Gaussian for small and moderate x, where it follows the tent-shaped exponential (also called double-exponential) Laplace law: $P_{\Delta t}(x) \propto \exp(-c|x|/\sqrt{\Delta t})$, as emphasized in Ref. [3]. The exponential distribution was found by many researchers [4–11], so it should be treated as a ubiquitous stylized fact for financial markets [3].

In order to explain the non-Gaussian character of the distribution of returns, models with stochastic volatility were proposed in literature [12–15]. If the variance v_t changes in time, then $v\Delta t$ in the Gaussian distribution should be replaced by the integrated variance $V_{\Delta t} = \int_{t_1}^{t_2} v_t \, dt$. If the variance is stochastic, then we

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should average over the probability distribution $Q_{\Delta t}(V)$ of the integrated variance V for the time interval Δt:

$$P_{\Delta t}(x) = \int_0^\infty \mathrm{d}V \, \frac{\mathrm{e}^{-x^2/2V}}{\sqrt{2\pi V}} Q_{\Delta t}(V). \tag{1}$$

The representation (1) is called the subordination [16,17]. In this approach, the non-Gaussian character of $P_{\Delta t}(x)$ results from a non-trivial distribution $Q_{\Delta t}(V)$.

In the models with stochastic volatility, the variables v or V are treated as hidden stochastic variables. One may try to identify these phenomenological variables with some empirically observable and measurable components of the financial data. It was argued [18–20] that the integrated variance $V_{\Delta t}$ may correspond to the number of trades (transactions) $N_{\Delta t}$ during the time interval Δt: $V_{\Delta t} = \xi N_{\Delta t}$, where ξ is a coefficient [21]. Every transaction may change the price up or down, so the probability distribution $P_N(x)$ after N trades would be Gaussian:

$$P_N(x) = \frac{\mathrm{e}^{-x^2/2\xi N}}{\sqrt{2\pi\xi N}}. \tag{2}$$

Then, the subordinated representation (1) becomes

$$P_{\Delta t}(x) = \int_0^\infty \mathrm{d}N \frac{\mathrm{e}^{-x^2/2\xi N}}{\sqrt{2\pi\xi N}} K_{\Delta t}(N), \tag{3}$$

where $K_{\Delta t}(N)$ is the probability to have N trades during the time interval Δt. (We assume that N is large and use integration, rather than summation, over N). In this approach, the stochastic variance v reflects the fluctuating rate of trading in the market.

Performing the Fourier transform of (3) with respect to x, we find that the characteristic function $\tilde{P}_{\Delta t}(k_x)$ is directly related to the Laplace transform $\tilde{K}_{\Delta t}(k_N)$ of $K_{\Delta t}(N)$ with respect to N, where k_x and k_N are the Fourier and Laplace variables conjugated to x and N:

$$\tilde{P}_{\Delta t}(k_x) = \int_0^\infty \mathrm{d}N \, \mathrm{e}^{-N\xi k_x^2/2} K_{\Delta t}(N) = \tilde{K}_{\Delta t}(\xi k_x^2/2). \tag{4}$$

In this paper, we study whether the subordinated representation (3) agrees with financial data. First, we verify whether $P_N(x)$ is Gaussian, as suggested by Eq. (2). Second, we check whether empirical data satisfy Eq. (4). Third, we obtain $K_{\Delta t}(N)$ empirically and, finally, discuss whether $P_{\Delta t}(x)$ constructed from Eq. (3) agrees with the data. Refs. [19,20] have already presented evidence in favor of the first conjecture; however, the other questions were not studied systematically in literature.

The subordination was also studied in physics literature as the continuous-time random walk (CTRW) [22,23]. Refs. [24–26] focused on the probability distribution of the waiting time Δt between two consecutive transactions ($\Delta N = 1$). Our approach is to study the distribution function $K_{\Delta t}(N)$, which gives complementary information and can be examined for a wide variety of time lags. In Ref. [27], this function was studied for some Russian stocks.

We use the TAQ database from NYSE [28], which records every transaction in the market (tick-by-tick data). We focus on the Intel stock (INTC), because it is highly traded, with the average number of transactions per day about 2.5×10^4. Here we present the data for the period 1 January–31 December 1999, but we found similar results for 1997 as well [29]. Because of difficulties in dealing with overnight price changes, we limit our consideration to the intraday data. Since Δt is relatively short here, the term $\mu\Delta t$ is small and can be neglected.

2. Probability distribution of log-returns x after N trades

It follows from Eq. (2) that $\langle x^2\rangle_N = \xi N$, where $\langle x^2\rangle_N$ is the second moment of x after N trades. It is also natural to expect that the average number of trades $\langle N\rangle_{\Delta t}$ during the time interval Δt is proportional to Δt with some coefficient η. Thus, we expect

$$\langle x^2\rangle_N = \xi N, \quad \langle N\rangle_{\Delta t} = \eta\Delta t, \quad \langle x^2\rangle_{\Delta t} = \theta\Delta t, \quad \theta = \xi\eta. \tag{5}$$