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Periodic attractors of random truncator maps

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Abstract

This paper introduces the *truncator* map as a dynamical system on the space of configurations of an interacting particle system. We represent the symbolic dynamics generated by this system as a non-commutative algebra and classify its periodic orbits using properties of endomorphisms of the resulting algebraic structure. A stochastic model is constructed on these endomorphisms, which leads to the classification of the distribution of periodic orbits for random truncator maps. This framework is applied to investigate the periodic transitions of Bornholdt's spin market model.

Keywords: Noncommutative algebraic dynamics; Iterated function systems; Periodic orbits; Stochastic endomorphisms

1. Introduction

Many situations in behavioral economics and finance involve the evolution of binary opinion allocations across a set of interacting agents. In this paper we define a particular type of dynamic on such opinion configurations, called the truncator map, and study its recurrence properties. The proposed framework describes the dynamics as an algebraic object, where the interaction potential is represented as the action of one configuarion on another. In this manner, opinion allocations adjust to one another and ultimately converge to various attractors. Our goal here is to characterize the periodic orbits of such opinion dynamics.

We proceed to define a Markov chain and represent the periodic orbits as solutions to first passage problems for the Markov generators. Throughout the paper, the algebraic structures we introduce help classify the interaction potentials that in turn define different models of opinion evolution. We begin by introducing the truncator map and relevant notation, and proceed to describe the ring structure underlying our model. The following section presents some partial results in an attempt to solve polynomial equations in the *opinion ring*. In the next section we randomize the interaction potential to arrive at a Markov chain representation of the dynamics, while the last section specializes this approach to a concrete stochastic market microstructure model.

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2. Model description

Let $\Omega = [-1,1]^N$ for some positive dimension N and consider a set $\{S_j\}_{j=1}^M$ of mutually exclusive and exhaustive subsets of Ω. A typical example will be the generalized quadrants, i.e.,

$$S_j = \left\{ x \in \Omega | \mathrm{sgn}(x_i) = \alpha_i, 1 \leqslant i \leqslant N \text{ and } \sum_{i=0}^{N-1} \alpha_{i+1} 2^i = 2^N + 1 - 2j \right\},$$

where the unique set $\{(1-\alpha_N)/2, (1-\alpha_{N-1})/2, \ldots, (1-\alpha_1)/2\}$ denotes the binary decomposition of the integer $j-1 < M$.

Given a mapping $f : \Omega \longrightarrow \Omega$, we define the *truncator* map as the following discrete dynamical system:

$$x(n+1)_i = x(n)_i \, \mathrm{sgn}(f(x(n))_i), \tag{2.1}$$

where $\mathrm{sgn}\, x = \lim_{k\to\infty} \tanh(kx)$. In this paper, we specialize to the case of *shuffling* maps, i.e., f which can be expressed as a set of invertible operators A_j associated with each component S_j of Ω.

Specifically, consider the finite group [15] $G = \{1, 2, \ldots, M\}$ endowed with a commutative operation $\circ$ such that, for every $g \in G$, $g \circ g = 1$. This group is naturally isomorphic to the cyclic product group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ of M factors, which can be represented as a modulo multiplication group M_n for some large enough n such that $\phi(n) = M$, where ϕ is the Euler totient function. In this setting, assign an orientation reversing invertible ℓ_∞ isometry A_j to each component S_j of Ω, with the property that $A_j(S_i) = S_{i\circ j}$. The associated *shuffling* map is given by a mapping $\varphi : G \longrightarrow G$ such that $f|_{S_i} = A_{\varphi(i)}$. Using this notation, the resulting truncator dynamics can be described as

$$x(n+1) = \sum_{i=1}^{M} A_{\varphi(i)}(x(n)) \mathbf{1}_{S_i}(x(n)). \tag{2.2}$$

These dynamics arise in a variety of settings [1–3]. We were driven to study the truncator dynamics because they represent the frozen phase limit ($\beta \to \infty$) of a class of interacting particle systems describing economic interactions and opinion formation [4–7]. In this setting, the points x represent configurations of a spin network and the shuffling map represents the interaction Hamiltonian that describes the influence of local and global effects to the flipping of individual spins.

Another setting where such truncator dynamics arise is that of random Boolean networks [16,17]. Often such models are used to describe regulatory networks (e.g. genetic or metabolic networks in biology [8–10,18]) and they are also used to describe instances of the satisfiability problem [11]. In this latter setting, global optimization algorithms are constructed to flip the values of Boolean variables populating the nodes of a graph in such a way as to maximize the probability that the clauses represented by the graph connections are simultaneously satisfied.

Our goal in this paper is to characterize the periodic attractors of the truncator map. Specifically we consider random endomorphisms of G [12,13] and derive the distribution of periodic orbits of the resulting random truncator dynamics. Of course the full truncator map (2.1) is generically chaotic [14], because there is sensitivity to initial conditions in the neighborhood of the boundaries between the components S_j (e.g. the axes, when the components are generalized quadrants). Here we will restrict our attention to shuffling maps and the resulting restricted truncator dynamics (2.2) which captures the spectrum of periodic attractors. In a later step we plan to use this analysis as a building block for understanding the transitions between the basins of attraction of the periodic attractors we describe here.

3. Algebraic dynamics

In order to better describe the orbits of (2.2) we define a new, non-commutative operation on G. This operation encodes the action of the shuffling map φ on G:

$$g_1 * g_2 = g_1 \circ \varphi(g_2). \tag{3.1}$$