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Dislocation pile-ups, slip-bands, ellipsoids, and cracks

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Abstract

The classic theories of dislocation pile-ups, initiated by Eshelby, Frank and Nabarro, and by Leibfried, can be greatly simplified if it is recognised that the dislocations in the pile-up will experience uniform stress if they are lodged in an ellipsoidal interface. Elementary algebra then produces the familiar results from continuum theory. It seems possible that the ellipsoid construction may represent physical reality if it is taken to represent a three-dimensional slip-band. If so, there are concentrated forces spreading the band perpendicular to the slip band as well as parallel to it. Such ellipsoids also represent Mode II and Mode III cracks, and give a method for dealing with the more complicated Mode I cracks.

Keywords: Dislocations; Pile-ups; Slip-bands; Ellipsoids

1. Eshelby's theorem

Eshelby [1] famously proved that the stress field inside an ellipsoidal inclusion which has undergone a uniform transformation strain is uniform. This is true for any linearly elastic body: it need not be elastically isotropic [2–4]. More recently, in a series of papers, Markenscoff [5], has proved essentially the inverse theorem: if, in a region contained within an internally stressed body there is a uniform stress, then the region must be ellipsoidal in shape. The proof, which is couched in mathematical terminology, can perhaps be summarised as follows: (i) Both strain and stress are derived from gradients of elastic displacement. Thus, elastic displacement can be regarded as a kind of potential for stress and strain. If the displacement varies linearly with position, both stress and strain are constant. (ii) At the interface between the inclusion and the matrix, the displacement must be continuous. This places a constraint on the shape of the interface. For example, suppose the inclusion were a cube with displacements parallel to one side. Then at a corner the normal component of the displacement would be discontinuous: it would not be possible there to match the inner and outer fields without building in a singularity such as a dislocation, which would introduce long range spatially varying stresses leaking into the cube. (iii) Consider just the normal component of the displacement at the interface. If the normal to the interface varies other than linearly with position, then in the interface the displacement too will not vary linearly, and it will not be possible to match it to a uniform internal strain field. (iv) A linearly varying surface normal is produced only by a quadratic surface, and the only closed quadratic surface is an ellipsoid.

The proof is thus very subtle, and requires a distinction to be made between the type of interfacial singularity introduced by continuous distributions of dislocations and that by single dislocations with isolated cores. However, it seems clear enough that the ellipsoid plays a unique role in inclusion problems, rather like the elliptical planetary orbit in Newtonian physics.

2. Slip bands, pile-ups, and ellipsoids

It is widely agreed that slip bands form suddenly and spread quickly, in a time of the order of microseconds. Slip bands require the co-ordinated motion of several dislocations on neighbouring slip planes. A plausible picture is that in a band the multiplication of glide dislocations is accompanied by their rapid motion under the influence of the applied stress.

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However, the freely slipping dislocations run into obstacles, on which they exert a highly concentrated force, proportional to their number. They also cause stress to build up on secondary slip systems, which if activated, produce a forest of obstacles. The forest of secondary dislocations acts like an almost instantaneously produced friction stress. They harden the slip band and relieve the stress.

If, at the moment when the glide dislocations are brought to a halt, one can think of them as experiencing a constant stress, namely zero, then the Eshelby theorem states that they must form an ellipsoidal inclusion. Traditionally, based on the work of Eshelby et al. [6], the halted dislocations are treated as a planar pile-up in which the force on each dislocation in the pile-up is zero because the force due to the applied stress is balanced by the stresses from other dislocations in the pile-up. A planar pile-up can be regarded as an ellipsoidal inclusion with zero aspect ratio. Only in an ellipsoidal slip band can the uniform stress resulting from the plastic strain be made to balance the applied stress. The notion that the slip band can be treated as an ellipsoidal inclusion in three dimensions is, thus, a generalisation of the earlier two-dimensional models.

Fig. 1 shows an ellipsoidal inclusion which we imagine to be a model of a freshly formed slip band on the *x*–*y* plane. The glide dislocations of Burgers vector *b* may be imagined to have a uniform spacing *h* perpendicular to the crystallographic slip planes. The plastic engineering shear strain in the band e_{13}^{P} is twice the tensor strain $\varepsilon_{13}^{\mathrm{P}}$ and is given by

$$e_{13}^{\mathrm{P}} = 2\varepsilon_{13}^{\mathrm{P}} = \frac{b}{h}. \quad (1)$$

These glide dislocations are 'geometrically necessary dislocations' which bound the slipped region of the glide band. On a scale greater than the separation of the crystallographic slip planes, they produce a uniform plastic shear, so if the inclusion is ellipsoidal the stress they produce in the interface is uniform and can be made equal and opposite to the applied stress.

We can now use the elementary properties of the ellipse in Fig. 1 to calculate the dislocation distribution. The equation

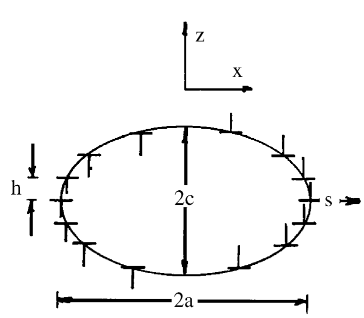

Fig. 1. An ellipsoidal slip band, semi-axes *a* and *c*, confined by obstacles. The dislocations are spaced uniformly in the vertical axis by a distance *h*. To calculate the escape pressure on the obstacles, all the dislocations are imagined to move an infinitesimal distance δa. The concentrated stress resulting from the fixed pile-up is calculated as a function of the distance *s*, measured from the tip of the band.

of the ellipse is $(x/a)^2 + (z/c)^2 = 1$. The number of dislocations per unit length in the direction perpendicular to the slip band, $n(z)$, is constant, namely $1/h$. The distribution projected onto the *a*-axis of the ellipse is given by

$$n(x) = 2n(z)\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2cx}{ha\sqrt{a^2 - x^2}}. \quad (2)$$

This gives us the dislocation distribution. The total number of dislocations on one side of the ellipse is

$$N = \frac{2c}{h}. \quad (3)$$

All that remains now is to calculate the relationship between the number of dislocations and the applied stress. To do this, we have recourse to the Eshelby model. In the constrained shear band, one which is embedded in the matrix which contains the obstacles to its propagation, there is uniformly zero shear stress, thus,

$$\sigma^{\mathrm{A}} + \sigma^{\mathrm{C}} = \sigma^{\mathrm{A}} - 2\mu\gamma\varepsilon^{\mathrm{P}} = 0. \quad (4)$$

That is, the sum of the applied stress, σ^{A}, plus the stress resulting from the change in shape of the ellipsoid, the constrained stress, σ^{C}, is zero. Values of the Eshelby accommodation factor γ which relates the transformation strain to the constrained stress can be found in the paper by Brown and Clarke [7]. Three cases of interest can be treated without difficulty: pile-ups of straight edge dislocations, pile-ups of straight screw dislocations, and pile-ups of circular loops against a circular obstacle. For straight dislocations, we are concerned with ribbon-shaped inclusions of elliptical cross-section, and for circular pile-ups we are concerned with penny-shaped inclusions, of radius *R*. In the case of edge dislocations, $\gamma = c/(1-\nu)a$. For screw dislocations, the factor $(1-\nu)$ is missing; for penny-shaped inclusions there is a more complicated expression. If these values are inserted into Eqs. (4) and (1) is used for the plastic strain, one finds the total number of dislocations as a function of the applied stress. The results are:

$$N^{\mathrm{EDGE}} = \frac{2a(1-\nu)\sigma^{\mathrm{A}}}{\mu b}, \qquad N^{\mathrm{SCREW}} = \frac{2a\sigma^{\mathrm{A}}}{\mu b}, \quad \text{and}$$

$$N^{\mathrm{CIRCULAR}} = \frac{8R(1-\nu)\sigma^{\mathrm{A}}}{\pi(2-\nu)\mu b}. \quad (5)$$

These are the standard results for planar pile-ups, as first obtained by Leibfried [8] and presented by many other authors, see for example, Hirth and Lothe [9]. What is new here is that the ellipsoidal pile-up gives identical results, provided the aspect ratio *c*/*a* is small enough that powers higher than one can be neglected. The argument depends only on the assumption that there are several notional slip planes parallel to the major axis of the ellipse. But if there is only one, and the slip band degenerates into a planar pile-up, the resulting formulae are still correct. In the ellipsoidal slip band, the number of dislocations is conveniently thought of as the