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A unified approach to portfolio selection in a tracking error framework with additional constraints on risk

Patrizia Stucchi*

Department of Economics and Statistics, University of Udine, Via Tomadini, 30, 33100 Udine (UD), Italy

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ABSTRACT

Most methods of performance evaluation and most allocation strategies are based on tracking error, that is the excess return of the managed portfolio with respect to the benchmark return. Analysis of the tracking error in a mean-variance framework has been performed by Roll (1992) who also investigated the impact of additional *beta* constraints, while Jorion (2003) considers constraints on total risk (portfolio variance). Alexander and Baptista (2010) add a constraint on the *alpha* of minimum tracking error variance portfolios. In other recent works, Alexander and Baptista (2008) and Palomba and Riccetti (2012) analyze the problem with *Value at Risk* (*VaR*) constraints under return normality assumption. This paper investigates the relationships between all these different approaches and provides a unified treatment. Moreover, analysis of the frontier of *Conditional VaR* constrained tracking error variance has been performed.

1. Introduction

Fund managers investment strategies are usually assessed with respect to a benchmark portfolio. The optimal portfolio may be selected in terms of its performance relative to the pre-specified benchmark. The key random variable in this process is the tracking error, that is the difference between the managed portfolio and benchmark returns.

Roll (1992) performs a detailed analysis of tracking error in a mean-variance framework *á la* Markowitz (1952, 1959). In his paper Roll develops a minimum tracking error variance (*TEV*) frontier where the risk is represented by the variance of the tracking error and the profitability by its expected return. The control variables are the differences (extra-weights), asset by asset, between the allocation weights of the managed portfolio and those of the benchmark.

In his paper, Roll investigates also the impact of additional constraints on β_{PB}, that is the *beta* of the managed portfolio *P* with respect to the benchmark *B*, in order to introduce further control of excessive risk. His analysis, focusing on a specific set of values of *beta*, shows that further constraints on *beta* may improve the choice when compared with unconstrained minimum *TEV* portfolios. Roll provides an enlightening explanation of this result, which is due to the inefficiency of the benchmark from the Markowitz classical point of view. An increasing degree of benchmark inefficiency results in an increasing distance between the classical and the *TEV* frontiers. The additional *beta* constraint suggested by Roll may be considered a pioneering idea consistent with practice: sponsors often ask the manager for specified levels of global portfolio risk and β_{PB} is a global portfolio risk measure, even though depending on the benchmark.

Jorion (2003) reinforces this point of view and suggests the use of constraints on the whole portfolio variance instead of *beta* constraints, with the purpose of controlling in some way the whole effective "absolute" riskiness of the managed portfolio.

Alexander and Baptista (2010) consider a *TEV* frontier adding constraints on *ex-ante alpha* obtaining an *alpha-TEV* frontier. They find that any portfolio on the *alpha-TEV* frontier also belongs to the *beta*-constrained mean-*TEV* frontier for some *beta* constraint and level of expected return.

* Tel.: +39 0432 249326; fax: +39 0432 249329.
E-mail address: patrizia.stucchi@uniud.it

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In other recent works, Alexander and Baptista (2008) and Palomba and Riccetti (2012) analyze the impact of adding a *Value at Risk* (*VaR*) constraint to the problem of an active manager who seeks to outperform a benchmark with the aim to control both *VaR* and *TEV*. Their framework is based on normally distributed returns.

The main purpose of my paper consists in the analysis of the relationships between the aforementioned approaches. In fact, if different choices for the risk measure constraint give rise to different responses in terms of the optimal portfolio composition or lead to different portfolio frontiers, the fund manager could have trouble explaining these conflicts to his sponsors. Obviously, the puzzle comes to a solution if there exists a unique optimal strategy or frontier under different additional risk constraints. This could be crucial for manager decisions and disclosure to the fund sponsors.

My second analysis starts from the idea that *VaR* is the risk measure of choice in risk management industry (see Alexander & Baptista, 2008; Palomba & Riccetti, 2012), even though it lacks the subadditivity property (as shown by Artzner, Delbaen, Eber, & Heath, 1999). Under regular distributions (e.g. normality) *Conditional Value at Risk* (*CVaR*) measure allows to overcome this difficulty. Moreover, *CVaR* takes into account particularly adverse financial markets conditions. In this paper, a detailed analysis of the *TEV* frontier with additional constraints on *CVaR* has been performed, included a comparison with other risk constrained *TEV* frontiers.

The paper is organized as follows. In Section 2 there is a brief recall of the main results obtained by Roll (1992). Section 3 describes in detail his *beta*-constrained model, whose relationship with the Jorion model is investigated in Section 4. Section 5 examines the *alpha-TEV* frontier obtained by Alexander and Baptista (2010). Section 6 is devoted to the analysis with *VaR* constraints whereas Section 7 to that with *CVaR* constraints. Numerical practical applications may be found in Section 8. Conclusions are drawn in Section 9. Proofs of the main results are shown in the Appendix.

2. The unconstrained minimum tracking error frontier

Roll (1992) defines the stochastic variable tracking error $\tilde{T}$ as the difference between the managed portfolio return $\tilde{R}_P$ and the benchmark return $\tilde{R}_B$:

$$\tilde{T} = \tilde{R}_P - \tilde{R}_B$$

The main assumption on fund manager behavior is that they want to minimize the tracking error variance for a given level of expected extra-performance with respect to the benchmark.

This means that the objective function is $V(\tilde{T})$:

$$V(\tilde{T}) = V(\tilde{R}_P - \tilde{R}_B) = \mathbf{q}'_P\mathbf{V}\mathbf{q}_P + \mathbf{q}'_B\mathbf{V}\mathbf{q}_B - 2\mathbf{q}'_P\mathbf{V}\mathbf{q}_B \tag{1}$$

where $\mathbf{q}_P$ denotes the vector of weights of the managed portfolio P, $\mathbf{q}_B$ the vector of weights of the benchmark B and $\mathbf{V}$ is the (definite positive) variance-covariance matrix of the n assets (stocks) in the market (or simply the universe of assets considered by the fund manager).

Eq. (1) is equivalent to:

$$V(\tilde{T}) = (\mathbf{q}_P - \mathbf{q}_B)'\mathbf{V}(\mathbf{q}_P - \mathbf{q}_B)$$

which allows us to express the variance $V(\tilde{T})$ as a function of the extra-weights $\mathbf{x} = \mathbf{q}_P - \mathbf{q}_B$:

$$V(\tilde{T}) = \mathbf{x}'\mathbf{V}\mathbf{x}$$

Let G be the expected extra-performance, the constraint on G in terms of $\mathbf{x}$ is:

$$G = E(\tilde{R}_P) - E(\tilde{R}_B) = E_P - E_B = \mathbf{x}'\mathbf{E}$$

with $\mathbf{E}$ the mean vector of the assets.

At the end, since P and the benchmark B must be both feasible portfolios, the following conditions must hold:

$$\mathbf{q}'_P\mathbf{1} = \mathbf{q}'_B\mathbf{1} = 1$$

with $\mathbf{1}$ the unit vector.

This provides the last constraint in terms of $\mathbf{x}$:

$$\mathbf{x}'\mathbf{1} = 0$$

Therefore the problem becomes:

$$\begin{cases} \min\limits_{\mathbf{x}} & V(\tilde{T}) = \mathbf{x}'\mathbf{V}\mathbf{x} \\ \textit{subject to} & \mathbf{x}'\mathbf{E} = G \\ & \mathbf{x}'\mathbf{1} = 0 \end{cases} \tag{2}$$

Roll (1992) shows that the solution in terms of the extra-weights is:

$$\mathbf{x} = \frac{cG\mathbf{V}^{-1}\mathbf{E} - bG\mathbf{V}^{-1}\mathbf{1}}{ac - b^2}$$

where

$$a = \mathbf{E}'\mathbf{V}^{-1}\mathbf{E},\ b = \mathbf{E}'\mathbf{V}^{-1}\mathbf{1} = \mathbf{1}'\mathbf{V}^{-1}\mathbf{E},\ c = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

Roll highlights that the vector $\mathbf{x}$ is a linear combination of the vectors of weights $\mathbf{q}_0$ and $\mathbf{q}_1$:

$$\mathbf{q}_0 = \frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} = \frac{\mathbf{V}^{-1}\mathbf{1}}{c}, \quad \mathbf{q}_1 = \frac{\mathbf{V}^{-1}\mathbf{E}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{E}} = \frac{\mathbf{V}^{-1}\mathbf{E}}{b}$$

This means that two-fund separation holds in Roll's framework as in the classical Markowitz framework (1952, 1959). The minimum tracking error variance frontier (*TEV* frontier) equation is:

$$V_P = V(\tilde{R}_P) = \mathbf{q}'_P\mathbf{V}\mathbf{q}_P = V_B + 2\mathbf{x}'\mathbf{V}\mathbf{q}_B + \mathbf{x}'\mathbf{V}\mathbf{x}$$

that is:

$$V_P = V_B + \frac{2(E_P - E_B)(cE_B - b)}{ac - b^2} + \frac{c(E_P - E_B)^2}{ac - b^2}$$

Let $d = (ac - b^2)/c$, $\Delta_1 = E_B - b/c$ (as in Jorion, 2003), then the equation of the *TEV* frontier becomes:

$$V_P = V_B + \frac{2\Delta_1(E_P - E_B)}{d} + \frac{(E_P - E_B)^2}{d}$$

or, in terms of G:

$$V_P = V_B + \frac{2\Delta_1 G + G^2}{d}$$

The equation of the classical Markowitz mean-variance frontier is:

$$V_P = \frac{cE_P^2 - 2bE + a}{ac - b^2} \tag{3}$$

with positive a, b and $ac - b^2$.

In terms of Jorion parameters, Eq. (3) may be rewritten in the following way:

$$V_P = \frac{(E_P - E_B)^2 + 2\Delta_1(E_P - E_B) + \Delta_1^2}{d} + \frac{1}{c} \tag{4}$$

that is, in terms of G:

$$V_P = \frac{(G + \Delta_1)^2}{d} + \frac{1}{c}$$

As can be easily observed, it seems that reasoning in terms of tracking error produces a "loss of efficiency" with respect to the