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On model specification and parameter space definitions in higher order spatial econometric models ☆

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ABSTRACT

Higher-order spatial econometric models that include more than one weights matrix have seen increasing use in the spatial econometrics literature. There are two distinct issues related to the specification of these extended models. The first issue is what form the higher-order spatial econometric model takes, i.e. higher-order polynomials in the spatial weights matrices vs. higher-order spatial autoregressive processes. The second issue relates to the parameter space in such models and how this can affect the choice of model specification, estimation, and inference. We outline a procedure that is simple both mathematically and computationally for finding the stationary region for spatial econometric models with up to K weights matrices for higher-order spatial autoregressive processes. We also compare and contrast this approach with the parameter space for models that incorporate higher-order polynomials in the spatial weights matrices. Regardless of the model utilized in empirical practice, ignoring the relevant parameter region can lead to incorrect inferences regarding both the nature of the spatial autocorrelation process and the effects of changes in covariates on the dependent variable.

1. Introduction

Higher-order spatial econometric models that include more than one weights matrix have seen increasing use in the spatial econometrics literature, e.g. Brandsma and Ketellapper (1979), Sherrell (1990), Hepple (1995), Bell and Bocksteal (2000), Bordignon et al. (2003), Lacombe (2004), Allers and Elhorst (2005), McMillen et al. (2007), Ward and Gleditsch (2008), Dall'Erba et al. (2008), Elhorst and Fréret (2009), Lee and Liu (2010), and Badinger and Egger (2011). These so-called higher order spatial econometric models allow for a richer dependence structure that is not capable of being captured in a standard single weights matrix spatial econometric model framework. Generally, there are two ways in which the dependence structure in these higher-order spatial econometric models can be incorporated. The first, which has seen the most application in applied studies, is a simple extension of the single-W spatial autoregressive model to the case of multiple-W weights matrix models, i.e. the higher-order spatial autoregressive model case. The second form, which has seen limited application in applied research, is the incorporation of higher-order polynomials in the spatial weights matrices. However, issues related to the appropriate parameter space in these multiple spatial weights matrix models, regardless of the form they take, have been either ignored or erroneously assumed to be simple extensions of the single weights matrix case. This may be due to mathematical and computational difficulties in determining the parameter space or to the presumption that certain forms of spatial autocorrelation (e.g. negative spatial autocorrelation) are unlikely to occur.

In this paper, we outline a simple procedure for finding the stationary region for models with multiple spatial weights matrices (i.e. $K \geq 2$) for the higher-order spatial autoregressive model case as well as the higher-order polynomials in the spatial weights matrices case. Finding the appropriate admissible parameter space is important for at least two reasons. First, recent advances in the literature have shown an increased interest in both the theoretical and empirical aspects of negative spatial autocorrelation, e.g. Griffith and Arbia (2010). In the present paper, we will show that negative spatial autocorrelation may also occur as a result of choosing two spatial weights matrices that partly overlap. Second, obtaining proper parameter estimates, which are based to some extent on identification of the proper parameter space, is required for making correct inferences regarding the effect of changes in independent variables on the

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dependent variable. These so-called "effects estimates", as developed by LeSage and Pace (2009), provide summary measures of the direct, indirect, and total effects and are conditional upon the values of the spatial autocorrelation parameters. We illustrate the different methods of calculating these effects estimates with a numerical example and show that regardless of the type of higher-order spatial econometric model utilized in applied practice, completely separating out the effects estimates for each individual weights matrix is not feasible.

2. First-order models

Consider a first-order spatial autoregressive process in which the variable y_i ($i=1,\ldots,N$) is regressed on the variable $\sum_j w_{ij} y_j$,

$$y_i = \delta \sum_{j=1}^{N} w_{ij} y_j + \varepsilon_i, \quad i = 1, \ldots, N \tag{1}$$

where δ is the spatial autoregressive coefficient, w_{ij} is the i,j-th element of the exogenous, non-negative $N \times N$ spatial weights matrix W with zero diagonal elements that describes the arrangement of the spatial units in the sample, and ε_i are i.i.d. innovations with zero mean and finite variance σ^2. In matrix notation, the same spatial autoregressive process can be rewritten as:

$$Y = \delta W Y + \varepsilon. \tag{2}$$

This model can be estimated by maximum likelihood (Ord, 1975), quasi-maximum likelihood (Lee, 2004), instrumental variables[1] (Kelejian and Prucha, 1998; Kelejian et al., 2004; Lee, 2003), generalized method of moments (Kelejian and Prucha, 1999), or by Bayesian Markov Chain Monte Carlo (MCMC) methods (LeSage, 1997). Deriving the asymptotic properties of the maximum likelihood and quasi-maximum likelihood estimator of model (1), Lee (2004) assumes that the matrix $I_N - \delta W$, where I_N is an identity matrix of order N, is nonsingular and that the row and column sums of W are uniformly bounded.[2] Kelejian and Prucha (1999) make the same assumption to ensure the (unique) definition of the vector Y in terms of the vector of innovations ε. For a symmetric W, the condition is satisfied as long as δ is in the interior of $(1/\omega_{\min},\ 1/\omega_{\max})$, where $\omega_{\min}$ and $\omega_{\max}$ denote, respectively, the smallest and largest real characteristic root of W. If W is then normalized, either by dividing the elements of each row by its row sum or by dividing all elements by the largest characteristic root, the parameter space becomes $(1/\omega_{\min},\ 1)$, since the largest characteristic root of W equals unity in this situation.[3] Things are more complicated when W is asymmetric before normalization. In fact, a non-symmetric spatial weights matrix may have complex characteristic roots. LeSage and Pace (2009, pp. 88–89) prove that in that case δ is restricted to the interval $(1/r_{\min},\ 1)$, where $r_{\min}$ equals the most negative purely real characteristic root of W after normalization.

Kelejian and Prucha (1999) assume that δ is restricted to the interval $(-1,\ 1)$.[4] Their assumption is based on earlier work (Kelejian and Robinson, 1995) demonstrating that the restriction $1/\omega_{\min} < \delta < 1/\omega_{\max}$ – before W is normalized – may be unnecessarily restrictive. This is because any first-order spatial autoregressive process is defined for every value of δ such that the matrix $(I_N - \delta W)$ is nonsingular. A simple example taken from Elhorst (2001) illustrates this. Let W be a binary spatial weights matrix for a sample of just two spatial units. As a result of this, $\omega_{\min} = -1$ and $\omega_{\max} = 1$. Additionally, if $\varepsilon \sim N(0, \sigma^2 I_N)$, then $Y \sim N\left(0, \frac{(1+\delta)}{(1-\delta^2)} \sigma^2 I_N\right)$. This shows that when the variance of ε is finite, the variance of titY is also finite as long as δ is different from $1/\omega_{\min}$ or $1/\omega_{\max}$ (see Kelejian and Prucha, 2010, for a generalization).

Bell and Bocksteal (2000) find it peculiar that the non-admissible values of δ are directly related to the eigenvalues of W that, in turn, depend on sample size. In other words, model (1) is characterized by a non-continuous parameter space that changes when new observations are added (or eliminated). To obtain a continuum of values, needed for the large sample theory to hold, Ord (1981) suggests to restrict δ to $1/\omega_{\min} < \delta < 1/\omega_{\max}$ before W is normalized and to $1/\omega_{\min} < \delta < 1$ after normalization. Similarly, Kelejian and Robinson (1995) suggest to restrict δ to $-1 < \delta < 1$ since values of δ smaller than or equal to -1 are very unlikely to occur in practice.

It should be noted that this last restriction emphasizes a similarity between time-series analysis and spatial econometrics. A first-order serial autoregressive process:

$$y_t = \rho y_{t-1} + \varepsilon_t, \tag{3}$$

with T observations is stationary if ρ lies in the interval $(-1,\ 1)$. However, the same interval for a first-order spatial autoregressive process would be too restrictive. For normalized spatial weights, the largest characteristic root is indeed $+1$, but no general result holds for the smallest characteristic root, and the lower bound will typically be less than -1.

3. Second-order models

While there might be some similarities for first-order models, substantive differences occur when considering second-order models. The time-series literature (see Beach and MacKinnon, 1978, and the references therein) has pointed out that a second-order serial autoregressive process:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t, \tag{4}$$

with T observations is stationary if $\rho_1 + \rho_2 < 1$, $1 + \rho_2 - \rho_1 > 0$ and $\rho_2 > -1$. These constraints define a triangular region with vertices at $(-2, -1)$, $(0, 1)$ and $(2, -1)$. Hamilton (1994, pp. 29–33) shows that a second-order serial autoregressive process $y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t$ can also be written as a second-order polynomial in the lag operator L, $(1 - \phi_1 L - \phi_2 L^2) Y_t = \epsilon_t$, provided that $\phi_1 = \rho_1 + \rho_2$ and $\phi_2 = -\rho_1 \rho_2$.

A second-order spatial autoregressive process takes the form:

$$Y = \delta_1 W_1 Y + \delta_2 W_2 Y + \varepsilon, \tag{5}$$

and a second-order polynomial in two spatial weights matrices W_1 and W_2:

$$(I_N - \lambda_1 W_1)(I_N - \lambda_2 W_2) Y = \varepsilon, \tag{6}$$

or, alternatively,

$$Y = \lambda_1 W_1 Y + \lambda_2 W_2 Y - \lambda_1 \lambda_2 W_1 W_2 Y + \varepsilon. \tag{7}$$

For simplicity, but without loss of generality, we assume that the spatial weights matrices W_1 and W_2 are normalized in the remainder of this paper.

The model in Eq. (6) "filters" the dependent variable vector Y for two types of spatial dependence, one reflected by the weights matrix W_1 and the other by W_2. Eq. (7) is a logical implication of this view of modeling spatial dependence and it implies that extending the first-order model to include more than one spatial weights matrix requires that we consider a cross-product term that appears as $W_1 W_2$. Eqs. (5)

[1] Kelejian and Prucha (1998) and Lee (2004) point out that the method of instrumental variables is ineffective when there are no regressors in the model.

[2] The maximum likelihood estimator also requires the error terms to be normally distributed, while the quasi-maximum likelihood estimator does not.

[3] As it is well known, a symmetric spatial weights matrix will not necessarily remain symmetric after row-standardization.

[4] See also Kelejian and Prucha (2002), Kelejian et al. (2004), Lee (2002, 2003) among others.