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Differential Geometry techniques in the Black-Scholes option pricing; theoretical results and approximations

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Abstract

Black-Scholes model for the basket options is used to valuate S & P 500, DAX and other Stock market index options. We explain the lack of closed formulas for the multi-asset European call options, using a differential geometric approach initiated by Labordere and V. Linetski. Our theoretical results can be tested on some extensions of the SABR model, Heston's and Bergomi stochastic volatility models, usually approached using Monte Carlo or numerical partial differential equation simulations.

Keywords: Black-Scholes option pricing model, local volatility, differential geometry, risk management, econophysics;

1. Introduction

Here introduce the paper, and put a nomenclature if necessary, in a box with the same font size as the rest of the paper. The paragraphs continue from here and are only separated by headings, subheadings, images and formulae. The section headings are arranged by numbers, bold and 10 pt. Here follows further instructions for authors.

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The evaluation of the European call options is based on generalized Black-Scholes models of local and stochastic volatility The stochastic differential equations which drive the assets imply that the option prices are solutions of second order partial differential equations. The economic reality and statistical techiques select a finite number of mathematical models, but the academic databases are poor in closed, specific formulas for the option pricing. We identify the cause of this lack of solvable models: it is not a result of a small number of local and stochastic volatility models. The coefficients of the Black-Scholes partial differential equations used in option pricing have to satisfy a specific system of differential equations, as a necessary condition to be solvable from the point of view of : path integral , differential geometry or stochastic calculus techiques. And of course it is unrealistic to expect that the assets are driven by a partial differential equation connected with basic concepts of Differential Geometry: Killing vector fields and the Ricci flow. In any case we can approximate the solutions of BSM equation using our approach. We also compute differential geometry parameters for the SABR, Heston and Bergomi volatility models, successfully applied on Wall Street and considered as representing four generations of option pricing models.

1.1. Differential Geometry and Option pricing

F **Henry-Labordere** uses a differential geometry approach: a heat kernel approximation in his study of Heston and SABR stochastic volatility models.

A path integral approach to option pricing was studied by **Linetzky** (1998) and **Taddei** (1999). They associate to any stochastic differential equation a Lagrangian functional and a Van Vleck determinant required to compute a path integral, which is computable in very few cases: Gaussian models and models that can be reduced to Gaussians by changes of variables, reparametrizations of time and projections (Ex: the Black–Scholes model, Ornstein–Uhlenbeck, Cox–Ingersoll–Ross model, Bessel process.)

Given an n-dimensional stochastic differential equation which describes the evolution of a basket of n asset prices, when does the option pricing Black-Scholes equation can be transformed to the heat equation in R^n **?** { W_i } are n independent standard Brownian motions. The risk neutral dynamics of n stocks is given by: $dX_t^\mu = rX_t^\mu dt + \sum_i \sigma_i^\mu (X(t),t) dW_i$. ***r*** is a constant interest rate. Define $G^{\alpha\beta}(x,t) = \sum_i \sigma_i^\alpha \sigma_i^\beta$

The multidimensional Black-Scholes option pricing equation is : $\partial_t f(x,t) + 1/2 G^{\alpha\beta} f_{\alpha\beta} = r\left(f - x_i \partial_i f\right)$

Then $h(x_1, x_2, \dots, x_n, t) = e^{-rt} f\left(x_1 e^{rt}, x_2 e^{rt}, \dots, x_n e^{rt}, t\right)$ satisfies $h_t + \frac{1}{2} g^{\alpha\beta} h_{\alpha\beta} = 0$ (2)

where $g^{\alpha\beta}(x,t) = G^{\alpha\beta}\left(xe^{rt}, t\right) e^{-2rt}$. $h_{\alpha\beta}$ are partial derivatives of h with respect to 2 variables ;summation understood

Definition: We say that the eq. **2** is equivalent to the heat equation if there are *n* functions

$H_1(x,t), \dots, H_n(x,t)$, t time and x in R^n such that for any *W*, solution of the equation **(3)** below,

$$W_t + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 W}{\partial x_i^2} = 0$$

, $W\left(H_1(x,t), \dots, H_n(x,t), t\right) = h(x,t)$ is a solution of **(2).** And for any *h* solution of (2), there is a *W* as above.

There is a path integral approach in computing certain financial and quantum mechanics observables. A path integral is a limit of a sequence of finite dimensional integrals. Also, Feynman –Kac formula connects the solution of the B.S. equation with probability theory . The transition probability function, or the Green kernel is computed as an integral over the *space of all paths*, where we assign a probability to each path.

$$f(t,x) = E^{t,x}\left[e^{-r(T-t)} h(X(T))\right] = \int dX(T) e^{-r(T-t)} h(X(T)) p(X(T), T \mid x(T), t)$$

$$O(S(t),t) = E^{t,S(t)}\left[e^{-r\tau} F\ (e^{X(T)})\right], \ \tau = T - t \quad [5]$$

Taddei and Linetzki() proposed a Lagrangian functional which can be used to approximate the Green kernel. Instantons are solutions of the Euler-Lagrange equation ,