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Short communication

Value at risk: Is a theoretically consistent axiomatic formulation possible?

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Abstract

This note identifies three properties of a risk measure, the acceptance of all of which implies the acceptance of the VaR risk measure; and the rejection of any one of which implies the rejection of the VaR risk measure. First, a risk measure should reflect weak aversion to losses. Second, only sufficiently likely threats matter. Finally, the risk measurement should be unaffected by how promising the upside may look like. These properties, by themselves, constitute a consistent set of axioms that are necessary and sufficient for the acceptance of the VaR risk measure on a *given* probability space. The axiomatization highlights a peculiar characteristic of VaR: it ignores the upside, while at the same time neglecting the worse of the downside.

Keywords: Value at risk; Coherent risk measures; Shackle; Downside risk; Expected shortfall

1. Introduction

Serious concerns have been raised about the possibility that VaR may fail to be subadditive.[1] This has fueled the development of alternative downside risk measures (see Artzner, Delbaen, Eber, & Heath, 1999; Acerbi & Tasche, 2002; Acerbi, 2004). Reflecting on the significance of the subadditivity debate, Acerbi, a leading critic of value at risk, writes:

> the main problem with VaR is not its lack of subadditivity, rather the very fact that no set of axioms for a risk measure and therefore no unambiguous definition of financial risk has ever

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[1] A risk measure ρ is subadditive if $\rho(X+Y) \leq \rho(X) + \rho(Y)$. It is through sub-additivity that a risk measure may reflect the risk-reducing benefits of diversification.

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> been associated to this statistics. So, despite the fact that some VaR supporters still claim that subadditivity is not a necessary axiom, none of them, to the best of our knowledge, has ever tried to write an alternative meaningful and consistent set of axioms for a risk measure which are fulfilled also by VaR. In other words, so far, we have never been told what concept of risk VaR has in mind. (Acerbi, 2004, p. 150).

I identify three properties of a risk measure, the acceptance of all of which implies the acceptance of the VaR risk measure; and the rejection of any one of which implies the rejection of the VaR risk measure. First, a risk measure should reflect weak aversion to losses. Second, only sufficiently likely threats matter. Finally, the risk measurement should be unaffected by how promising the upside may look like. These properties, by themselves, constitute a consistent set of axioms that are necessary and sufficient for the acceptance of the VaR risk measure on a *given* probability space.

1.1. Notation

$(\Omega, \Im, P)$	Probability space
$\mathbf{X}, \mathbf{Y} \in \boldsymbol{\beta}$	Real-valued random variables on $(\Omega, \Im, P)$
$\boldsymbol{\rho}_\alpha : \boldsymbol{\beta} \to \Re$	Risk measure with parameter $\boldsymbol{\alpha} \in (0, 1)$
$\mathbf{q}_\alpha[X] = \inf\{x \in \Re : P(X \leq x) \geq \alpha\}$	α-quantile of X; $\boldsymbol{\alpha} \in (0, 1)$ and $inf\emptyset = \infty$
$\mathbf{VaR}[X] = -\mathbf{q}_\alpha[X]$	Value-at-risk of X at confidence level $\boldsymbol{\alpha} \in (0, 1)$

2. Properties of a risk measure

There are three properties of a risk measure, the acceptance of all of which implies the acceptance of the VaR risk measure; and the rejection of any one of which implies the rejection of the VaR risk measure.

Property A. *Weak loss aversion: X is at least as risky as Y if the probability of a worse outcome than any given value is greater for X than for Y.*

$$If\ P(X < t) \geq P(Y < t) \quad for\ all \quad t \in \Re, \quad then \quad \rho_\alpha(X) \geq \rho_\alpha(Y) \tag{1}$$

Property B. *Only sufficiently likely threats matter: risk cannot be reduced by redistributing probability mass in the lower tail of a distribution, which collectively has a probability of occurence of less than any given $\alpha \in (0, 1)$. If Y can be derived from X by redistributing the probability mass in the lower α-quantile of the distribution of X, then Y cannot be less risky than X.*[2]

$$If\ P(X < t) = P(Y < t) \quad for \quad t > q_\alpha[X], \quad then \quad \rho_\alpha(X) \leq \rho_\alpha(Y) \tag{2}$$

Property C. *The upside risk does not matter: risk cannot be increased by redistributing probability mass in the upper tail of a distribution, which collectively has a probability of occurence of less than $(1-\alpha)$, for any given $\alpha \in (0, 1)$. If Y can be derived from X by redistributing the*

[2] For example, suppose $\alpha = 0.01$. Let X be a discrete random variable with distribution $P(X = -10000) = 0.009$, $P(X = -1000) = 0.001$ and $P(X = 1000) = 0.99$. Let Y be derived from X by redistributing the probability mass in the lower α-quantile of the distribution of X, such that Y has a distribution $P(Y = -1000) = 0.01$ and $P(Y = 1000) = 0.99$. Then, Property B asserts that Y cannot be less risky than X.