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Analytical solution of a multi-dimensional Hotelling model with quadratic transportation costs

Hernán Larralde [a], Juliette Stehlé [b,c], Pablo Jensen [b,c,d,e,*]

[a] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Apdo. Postal 48-3, C.P. 62251, Cuernavaca, Morelos, Mexico
[b] Université de Lyon, France
[c] Institut des Systèmes Complexes Rhône-Alpes (IXXI), France
[d] Laboratoire de Physique, École Normale Supérieure de Lyon and CNRS, 69007 Lyon, France
[e] Laboratoire d'Économie des Transports, Université Lyon 2 and CNRS, 69007 Lyon, France

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ABSTRACT

We present an analytical solution to the two-dimensional two-stage Hotelling model with quadratic transportation costs. We assume that consumers' choice is tempered by a logit function, which characterizes consumers' heterogeneity. As in the one-dimensional case, stores aggregate spatially when consumers' heterogeneity is strong enough. When it decreases, we show that stores differentiate in only one dimension. The analytical solution allows us to give a precise interpretation of this effect through the comparison of consumers' elasticity under differentiation along one or two characteristics. Finally, we extend our results to a hypercube of any dimension.

1. Introduction

Hotelling's model (Hotelling, 1929) is one of the preferred "toy models" of spatial economics. Variations of this model and its "Principle of Minimum Differentiation" have generated a large literature (for reviews, see (Anderson et al., 1992; Tirole, 1998; Hoover and Giarratani, 1984; Brown, 1989), for recent papers, see (Veendorp and Majeed, 1995; Tabuchi, 1994; Irmen and Thisse, 1998; Konishi, 2005)) allowing researchers to play with elementary mechanisms at will. In essence, this model consists in "consumers" that are distributed in a bounded space and choose to buy at the store which maximizes their utility. The definition of the consumer utility includes store prices and transportation costs. The aim of the model is to determine optimal locations and prices for competing stores trying to maximize their own profits.

The main conclusions of the one dimensional Hotelling model for two stores looking for "location then prices" optima are summarized in (Anderson et al., 1992). The locations chosen result from a tension between a "competition effect" (or market stealing effect) and a market power effect. On the one hand a firm wants to be near the market center in order to steal customers from its rival. On the other hand, a firm wants to be distant from its rival in order to soften price competition. The relative importance of both factors – and therefore the optimal locations – depends on consumers' heterogeneity. This heterogeneity represents additional choice factors, not taken into account explicitly in the analysis, which enter the picture as a random "noise" in the utility functions. Usually, this effect is introduced through a logit function (Anderson et al., 1992), the heterogeneity magnitude being quantified by the parameter μ. When the heterogeneity is great enough, the market stealing effect is dominant: consumers view the two stores as being sufficiently different that price competition between them is muted even when they are close to each other. In contrast, when heterogeneity is small, the market power effect is dominant, overcoming the positive impact that a unilateral move towards the market center has on demand. Therefore, stores prefer to separate in order to soften price competition.

The original one dimensional Hotelling model has been extended to two dimensional spaces by Veendorp and Majeed (1995) and Tabuchi (1994). These studies have shown that, when consumers' heterogeneity is not taken into account (μ=0), stores maximally differentiate in *one* of the dimensions, while adopting an identical location in the other dimension. Irmen and Thisse (1998) have generalized this result to any dimensions, imposing the condition that one characteristic is dominant in the consumer's choice.

To the best of our knowledge, there have been no general studies including consumers' heterogeneity ($\mu \neq 0$) in more than one dimension. In the present paper, we present an analytical solution to a Hotelling model with quadratic transportation costs, for arbitrary values of the heterogeneity parameter μ. We show that, when μ becomes smaller than a precise threshold, stores differentiate in only *one* dimension, without assuming any condition on the weight of the different characteristics. The analytical solution allows us to give a precise interpretation of this effect, in terms of a smaller elasticity

* Corresponding author.
E-mail address: pablo.jensen@ens-lyon.fr (P. Jensen).

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when the differentiation occurs along a single dimension. Finally, we generalize some of our results to any number of dimensions. Namely, we show that, independently of the number of dimensions, when μ becomes smaller than a precise threshold, stores prefer to start differentiating in only *one* dimension. This strongly suggests that minimal differentiation is the rule in any dimension.

2. Optimal store locations in a two dimensional city

2.1. Notation

For our analysis we consider a two-dimensional square city of side length L occupied uniformly by consumers with unit density.

We model consumers' behavior by the representative consumer's utility, which can be separated into two parts: a deterministic part including price and distance, and a stochastic one which characterizes the consumer's heterogeneity. The deterministic contribution to the utility of the representative consumer at position J for store S is $K_{J,S}$ given by:

$$K_{J,S} = R - p_S - ad^2_{J,S} \tag{1}$$

where R is the maximum utility of buying the product, assumed to be high enough to prevent any negative value for $K_{J,S}$, p_S is the price of the product at store S, $d_{J,S}$ the (Euclidean) distance between a consumer at J and the store at S, and a is the transportation cost coefficient.

Note that the store positions S_1 and S_2, as well as the consumer's positions J are vectors (i.e. $J=(j_x,j_y)$, etc); and, for simplicity of calculation, in our analysis we assume these to be continuous variables.

The stochastic contribution to the consumer's utility represents other factors that enter the choice process. Using the standard “logit” model (Anderson et al., 1992), we therefore assign a probability for a consumer at position J to buy at the store located at S as:

$$\delta_{J,S} = \frac{\exp(K_{J,S}/\mu)}{\sum_{S'} \exp(K_{J,S'}/\mu)} \tag{2}$$

where μ is the parameter which defines how sharply consumers discern between the deterministic utilities offered by each site. As $\mu \to \infty$ consumers do not discriminate between eligible stores, whereas if $\mu \to 0$, consumers exclusively choose the store with the highest $K_{J,S}$, taking only prices and distances into account.

In this situation, the average market D_S for the store at S is given by

$$D_S = \sum_J \delta_{J,S} \tag{3}$$

and the expected profit accrued by this store will be given by:

$$\Pi_S = p_S D_S \tag{4}$$

2.2. Analytical solution

For definiteness, we list here the precise rules of the system under consideration:

- First, given any two positions of the stores, S_1 and S_2, the stores compete in prices until they reach (Nash) equilibrium; that is, until any unilateral change in price leads to a lower profit for either store (stores are not allowed to cooperate). We further assume that stores know the profits accrued by the equilibrium prices for every pair of store positions.
- Knowing the equilibrium prices for each pair of sites and the position of its competitor, stores then compete for optimal (maximal profit) location, until they reach (Nash) equilibrium in position.

Now, since we have assumed a unit density of consumers, the average markets (number of consumers) D_1 and D_2, attending store 1 and store 2 respectively, will be given by:

$$D_1 = \int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y \frac{1}{1 + e^{\{p_1 - p_2 + a[|J-S_1|^2 - |J-S_2|^2]\}/\mu}} \quad \text{and} \quad D_2 = L^2 - D_1 \tag{5}$$

Once the locations S_1 and S_2 are given, price equilibrium is attained when

$$\left.\frac{\partial \Pi_1}{\partial p_1}\right|_{p_2} = 0 \quad \text{and} \quad \left.\frac{\partial \Pi_2}{\partial p_2}\right|_{p_1} = 0 \tag{6}$$

i.e. when any unilateral change in the price offered by either store leads to a reduction of the profit gained by that store (of course, the above equations only reflect that at equilibrium profit is an extremum, that these extrema are maxima was checked by numerical simulations (see below)).

The above are two coupled equations for the prices, from which we can determine, in principle, the optimal prices as functions of the store positions.

Now, equilibrium in the location competition will be achieved when neither store can increase its profit by changing its own position given the other store's location. Specifically, as mentioned above, storeowners know that a change in position will lead to a new set of prices, and they can evaluate the resulting change in profit accrued in the new position at the equilibrium prices corresponding to the new positions.

Thus, equilibrium is achieved when

$$\left(\frac{d\Pi_1}{dx_1}, \frac{d\Pi_1}{dy_1}\right) = 0 \quad \text{and} \quad \left(\frac{d\Pi_2}{dx_2}, \frac{d\Pi_2}{dy_2}\right) = 0 \tag{7}$$

where (x_1,y_1) and (x_2,y_2) are, respectively, the components of S_1 and S_2. Note that we do not restrain equilibrium solutions to remain within the square of length L. Regarding notation we choose to write the above equations as vectors of total derivatives with respect to the components of the positions of each store and reserve the gradient as a vector of partial derivatives to be used below.

From Eq. (4), we can write the equation for price equilibrium for store 1 as:

$$p_1 \frac{\partial D_1}{\partial p_1} + D_1 = 0 \tag{8}$$

then, using Eq. (8), the condition for location equilibrium becomes

$$\left(\frac{d\Pi_1}{dx_1}, \frac{d\Pi_1}{dy_1}\right) = p_1 \frac{\partial D_1}{\partial p_2} \nabla_1 p_2 + p_1 \nabla_1 D_1 = 0 \tag{9}$$

where $\nabla_1 = (\partial/\partial x_1, \partial/\partial y_1)$. Note that, as required in the description of the location game, the above expression includes the change in p_2 due to the change of location of store 1. A corresponding equation holds for store 2:

$$\left(\frac{d\Pi_2}{dx_2}, \frac{d\Pi_2}{dy_2}\right) = p_2 \frac{\partial D_2}{\partial p_1} \nabla_2 p_1 + p_2 \nabla_2 D_2 = 0 \tag{10}$$

From the explicit expression of D_1, we have:

$$I \equiv \frac{\partial D_1}{\partial p_2} = \frac{1}{4\mu} \int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y \frac{1}{\cosh^2\left(\{p_1 - p_2 + a[|J-S_1|^2 - |J-S_2|^2]\}/2\mu\right)} \tag{11}$$

whereas

$$\nabla_1 D_1 = \frac{a}{2\mu} \int_{-L/2}^{L/2} dj_x \int_{-L/2}^{L/2} dj_y \frac{J - S_1}{\cosh^2\left(\{p_1 - p_2 + a[|J-S_1|^2 - |J-S_2|^2]\}/2\mu\right)} \tag{12}$$