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Exact commutation relations for the Cooper pair operators and the problem of two interacting Cooper's pairs

I.G. Kaplan *, O. Navarro, J.A. Sánchez

Instituto de Investigaciones en Materiales, Universidad Nacional Autónoma de México, Apartado, Postal 70-360, 04510, México, D.F., Mexico

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Abstract

The analysis of trilinear commutation relations for the Cooper pair operators reveals that they correspond to the modified parafermi statistics of rank $p = 1$. Two different expressions for the Cooper pair number operator are presented. We demonstrate that the calculations with a Hamiltonian expressed via pairon operators is more convenient using the commutation properties of these operators without presenting them as a product of fermion operators. This allows to study problems in which the interactions between Cooper's pairs are also included. The problem with two interacting Cooper's pairs is resolved and its generalization in the case of large systems is discussed.

Keywords: Cooper's pair commutation relations; Pairing interactions; Strongly correlated electron systems

1. Introduction

It is well-known that the theory of the low temperature superconductivity was created by Bardeen, Cooper, and Schrieffer (BCS) [1] only after Cooper [2] had shown that two electrons interacting above the Fermi sea of non-interacting electrons can couple in a stable pair, if the interaction resulting from virtual exchange of phonons is attractive near the Fermi surface. As was demonstrated in a more sophisticated study [3], in full agreement with the Cooper assumption, the largest binding energy of the Cooper pair corresponds to electrons with the opposite momenta and spins. In the second quantization formalism, the operators of creation, b_k^+, and annihilation, b_k, of Cooper's pair in a state $(\mathbf{k}\alpha, -\mathbf{k}\beta)$, are defined as simple products of the electron creation and annihilation operators, $c_{k\sigma}^+$ and $c_{k\sigma}$, satisfying the fermion commutation relations,

* Corresponding author.
E-mail address: kaplan@zinalco.iimatercu.unam.mx (I.G. Kaplan).

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$$b_k^+ = c_{k\alpha}^+ c_{-k\beta}^+,\qquad b_k = c_{-k\beta} c_{k\alpha}. \tag{1}$$

Let us call these operators, following Schrieffer [3], as 'pairon' operators.

The Cooper pair has the total spin $S=0$. Hence, in accordance with the Pauli principle, the wave functions describing the Cooper pair system have the boson permutation symmetry, that is, they are symmetric under permutations of pairs. But the pairon operators (1) do not obey the boson commutation relations [1,3]. It is easy to show by direct calculation. Namely,

$$[b_k, b_{k'}^+]_- = [b_k^+, b_{k'}^+]_- = [b_k, b_{k'}]_- = 0 \quad \text{for } k \neq k', \tag{2}$$

$$[b_k, b_k^+]_- = 1 - \hat{n}_{k\alpha} - \hat{n}_{-k\beta}, \tag{3}$$

$$(b_k^+)^2 = (b_k)^2 = 0, \tag{4}$$

where $\hat{n}_{k\alpha} = c_{k\sigma}^+ c_{k\sigma}$ is the electron number operator. As follows from Eqs. (2)–(4), for $k \neq k'$ the Cooper pairs are bosons, while for $k = k'$ they do not obey the boson commutation relations, although they obey the Pauli principle and have the fermion occupation numbers for one-particle states.

Thus, the pairon operators may not be considered neither as the Bose operators, nor as the Fermi operators. This is the reason that the problem with the model Hamiltonian of the BCS theory

$$H = \sum_k \varepsilon_k b_k^+ b_k + \sum_{k',k} V_{kk'} b_{k'}^+ b_k \tag{5}$$

cannot be directly solved by transforming Hamiltonian (5) to the diagonalized form

$$H = \sum_n \varepsilon_k' B_k^+ B_k \tag{6}$$

by means of some unitary transformation

$$B_n = \sum_k u_{nk} b_k, \quad B_n^+ = \sum_k u_{nk}^* b_k^+. \tag{7}$$

The unitary transformation is canonical only for the Bose or Fermi operators. In general case, it is not canonical; it does not preserve the commutation properties of the operators transformed. Therefore, practically all calculations in the BCS approach were performed using the fermion properties of electron operators forming the Cooper pair.

In this paper we analyze the commutation relations for the pairon operators and reveal that they correspond to the modified parafermi statistics of rank $p = 1$. The general expression for the Cooper pair number operator is analyzed and it is proved that the same expression as for the boson (fermion) number operator can be also used in the pairon case. We demonstrate that the calculations with a Hamiltonian expressed via the pairon operators more convenient to perform using the commutation properties of these operators without presenting them as a product of fermion operators. This allows to study problems in which the interactions between Cooper's pairs are also included. The solution of the simplest problem with two interacting Cooper's pairs is presented.

2. Statistics of Cooper's pairs

As was discussed in the introduction, the pairon operators, Eq. (1), obey boson commutation relations only in the case of different momenta. For equal momenta, the right-hand part of commutation relation (3) contains the products of fermion operators that reflects the fermion structure of pairon operators.

$$[b_k, b_k^+]_- = 1 - c_{k\alpha}^+ c_{k\alpha} - c_{-k\beta}^+ c_{-k\beta}. \tag{8}$$

To operate with the pairon operators, the commutation relations for these operators do not have to include other kinds of operators. One of the ways to achieve this goal is to calculate trilinear commutation relations, as it is formulated in the parastatistics [4], for a short description see Refs. [5,6].

The direct calculation leads to the following trilinear commutation relations

$$\left[[b_k^+, b_{k'}]_-, b_{k''}^+\right]_- = 2\delta_{kk'}\delta_{kk''} b_k^+, \tag{9}$$

$$\left[[b_k^+, b_{k'}]_-, b_{k''}\right]_- = -2\delta_{kk'}\delta_{kk''} b_k. \tag{10}$$

These relations coincide with the trilinear commutation relations of the parafermi statistics for $k = k' = k''$. For different k, k' and k'' the relations are different. In the parafermi statistics in relations corresponding to Eq. (9) instead of the two pre-