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On the boundary behavior of the excess demand function

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ABSTRACT

I revisit thoroughly the standard boundary behavior satisfied by the excess demand function of a competitive economy, and I prove two implications of the boundary behavior. Then, I show that these implications lead to an alternative proof of the existence of competitive equilibria which is instructive, shorter and perhaps easier than the available proofs in the literature.

1. Introduction

Consider a competitive economy, with strictly monotonic preferences, which generates an aggregate excess demand function. To keep the analysis simple, I will focus on excess demand functions.[1] Consider the canonical proof of the existence of a competitive equilibrium, for instance Mas-Colell et al. (1995, Proposition 17.C.1) or Aliprantis et al. (1990, Theorem 1.4.8). Despite its elegance and beauty, that proof is lengthy and therein the boundary behavior of the excess demand function comes into play to show that the properly defined fixed point correspondence is upper hemicontinuous. Hence, the question I ask is whether, retaining the standard assumptions, one can come up with a shorter proof where the role played by the boundary behavior stands out in a novel and easy way.

To address this question, in Section 2.1 I will prove a lemma which highlights an easy consequence of the boundary behavior of the aggregate excess demand function. To the best of my knowledge, such a consequence has been either overlooked, or neglected, or regarded as foregone. In short, given any sequence of prices converging to the boundary of the price-simplex, the boundary behavior implies that the value of the excess demand function must be positive for infinitely many elements of the given sequence. More importantly, I will prove a corollary which may be put as follows: if π is a price-vector such that the excess demand function belongs to the polar of a certain trimmed simplex, then π must lie in the relative interior of the trimmed simplex. Finally, this fact is used, in Section 3, to give a new proof of the existence of competitive equilibrium which is shorter and probably easier than the standard proofs mentioned above. The key tool employed in the proof is a theorem due to Yannelis and Prabhakar (1983). Moreover, the proof of existence I propose is perhaps quite interesting from a pedagogical standpoint, since my approach is related to the literature about the existence of maximal elements and variational inequality problems (see Section 4).

Even though it would be pretentious to overemphasize the scope of this paper, let me stress that there is a recent revival of interest in simple existence proofs. See, for instance, Quah (2008), Fraysse' (2009), and Mackowiak (2010). My paper fits well in this strand of the literature. In order to prove the existence of competitive equilibrium, notice that Quah (2008) and Fraysse' (2009) impose an additional assumption on the aggregate excess demand correspondence

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[1] All the arguments herein presented may be adapted to deal with an excess demand correspondence.

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(function), namely the weak axiom. In contrast, I do not make any additional assumptions. This is not to downplay Quah and Fraysse's contributions. In fact, Quah's method is applicable to other contexts (existence of maximal elements for non-transitive preferences), and Fraysse' demonstrates how the existence-of-equilibrium problem can be attacked looking for equilibrium prices of a two-good economy.

The paper is structured as follows: in Section 2, I sketch the underlying economy, and I remind the reader the properties satisfied by the aggregate excess demand function. The focus is, obviously, on the boundary behavior. In Section 2.1, I prove the lemma and its corollary mentioned above. In Section 3, I prove a theorem on the existence of a competitive equilibrium price vector. In Section 4, I discuss how this paper is related to the existing literature. Postponing the section 'relation to the literature' should help the reader assess the contribution of this paper.

2. Notation and the boundary behavior

I will be dealing with an excess demand function which is not defined on the boundary of the price-simplex. As it is well-known, this is the setting where a boundary condition comes into play.

Let $Z : \mathbb{R}^N_{++} \to \mathbb{R}^N$ be a function satisfying the following properties:

(i) Z is continuous on $\mathbb{R}^N_{++}$.
(ii) $Z(p) = Z(\lambda p)$ for all $p \in \mathbb{R}^N_{++}$ and all $\lambda > 0$.
(iii) $p \cdot Z(p) = 0$ for all $p \in \mathbb{R}^N_{++}$.
(iv) There exists a $s > 0$ such that $Z_i(p) > -s$ for all $p \in \mathbb{R}^N_{++}$ and all $i = 1, 2, \ldots, N$.
(v) If $p_n \to p$, where $p \neq 0$ and $p_i = 0$ for some i, then

$$\text{Max}\{Z_1(p_n), \ldots, Z_N(p_n)\} \to +\infty$$

Now recall the following fact (see, e.g., Mas-Colell et al., 1995; Aliprantis et al., 1990): any finite-dimensional economy with continuous, strictly convex and strictly monotonic preferences,[2] gives rise to an aggregate excess demand function enjoying the above properties. Clearly, a competitive equilibrium price vector is a $p^* \in \mathbb{R}^N_{++}$ such that $Z(p^*) = 0$.

Property (v) is the one I am especially interested in. It is the standard boundary behavior of the excess demand function. I will prove (see corollary below) a quite interesting result: loosely put, property (v) restricts the position of those price-vectors for which the value of the excess demand is not positive when evaluated at any strictly positive vector. This provides a simple formulation of the boundary behavior that no longer involves an asymptotic condition. As I show in the sequel, my corollary leads to a simple and short proof of the existence of a competitive equilibrium for economies in which the aggregate excess demand function is not defined on the boundary of the price-simplex.

Now I turn briefly to constant returns-to-scale production economies. Clearly, in this case the production set is neither strictly convex nor bounded above. I borrow the formalization of the economy and the definition of competitive equilibrium from Geanakoplos (2003).

To this end, consider an economy with preferences that are continuous, strictly convex and strictly monotonic. A constant returns-to scale economy is a pair (Z, Y), where $Z : \mathbb{R}^N_{++} \to \mathbb{R}^N$ is the (consumers) aggregate net demand function, and $Y \subset \mathbb{R}^N$ is a closed and convex cone that allows for free disposal. Under these assumptions, Z still satisfies properties (i) through (v) above. In this case, the price domain must be restricted to the set of $p \in \mathbb{R}^N_{++}$ such that $pY \leq 0$, i.e., $py \leq 0$ for all $y \in Y$. One can assume, without much loss in generality, that the set of $p \in \mathbb{R}^N_{++}$ such that $pY \leq 0$ is non-empty.[3] A competitive equilibrium for a constant returns-to-scale production economy can now be defined as a price $\overline{p} \in \mathbb{R}^N_{++}$ such that $Z(\overline{p}) \in Y$ and $\overline{p}Y \leq 0$.

Even though the definition of competitive equilibrium depends on the type of economy at hand, in the next sections I will develop a unifying approach that can handle both type of economies, general production economies and constant returns-to scale economies. I will dwell on the proof of the existence of competitive equilibria only for general production economies, as the approach here proposed may be supplemented by Geanakoplos' 'Technology Lemma' (Geanakoplos, 2003, p. 598), nearly verbatim, to analyze also constant returns-to-scale production economies.

A piece of notation is in order: let $\Delta = \{p \in \mathbb{R}^N_+ : p \cdot \mathbf{1} = 1\}$, where $\mathbf{1}$ is the N-dimensional vector $(1, 1, \ldots, 1)$, and let $Int\Delta = \{p \in \mathbb{R}^N_{++} : p \cdot \mathbf{1} = 1\}$. Let $\partial\Delta = \{p \in \Delta : p_i = 0$ for some $i = 1, 2, \ldots, N\}$. Also, for any $n \geq N$, let $\Delta_n = \{p \in \Delta : p_i \geq (1/n)$ for all $i = 1, 2, \ldots, N\}$.

Clearly, since I am searching for a $p^* \in \mathbb{R}^N_{++}$ such that $Z(p^*) = 0$, by virtue of property (ii) above one can restrict the domain of Z to $Int\Delta$.

2.1. A preliminary lemma

The following result is a straightforward consequence of the boundary behavior of the excess demand function (and of the assumption that the excess demand function is bounded below). I do not claim originality, but I have not been able to find a reference in the literature.

[2] And with production sets that are closed, strictly convex, bounded above, and such that a strictly positive aggregate consumption bundle is producible using the initial endowments.

[3] See, e.g., Geanakoplos (2003, p. 599).