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Nonlinear difference equations, bifurcations and chaos: An introduction

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Abstract

The aim of these lecture notes is to present a few mathematical facts about the bifurcations of nonlinear difference equations, in a concise and simple form that might be useable by economic theorists.

Keywords: Nonlinear dynamics; Bifurcations; Chaos; Business cycles

1. Preliminaries

We state first a few more or less elementary facts about matrices and differentiable maps, that are used repeatedly in this article.

We recall that $\mathbb{R}^m$ is the set of all m-tuples of real numbers. A "point" or a "vector" of $\mathbb{R}^m$ is $x = (x_1, \ldots, x_m)$; the number x_i is the ith *coordinate* of the vector. Vectors x, y are added coordinatewise

$$x + y = (x_1, \ldots, x_m) + (y_1, \ldots, y_m) = (x_1 + y_1, \ldots, x_m + y_m).$$

If α is a real number, the product αx is the vector $(\alpha x_1, \ldots, \alpha x_m)$. $\mathbb{R}^m$ is then an m-dimensional real vector space. Its *standard basis* is the collection of vectors $(e_1, \ldots, e_m)$, in which for each $i = 1, \ldots, m$, e_i is the vector of coordinates $e_{ij} = \delta_{ij}$, $j = 1, \ldots, m$, where δ_{ij} is the Kronecker function, that is $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$. Any vector $x = (x_1, \ldots, x_m)$ has then a unique representation as a linear combination of the vectors e_i of the standard basis, that is $x = \sum_i x_i e_i$. A *norm* is a real valued function $\|.\|$ defined on $\mathbb{R}^m$, with $\|x\| \geqq 0$, such that $\|\alpha x\| = |\alpha| \, \|x\|$, $\|x + y\| \leqq \|x\| + \|y\|$, and $\|x\| = 0$ if and only if $x = 0$. The *Euclidean norm* will be denoted $|x| = (\sum_i x_i^2)^{1/2}$.

1.1. Matrix algebra

A square *matrix* of dimension m, i.e. a collection of m^2 real numbers $A = [a_{ij}]$, where $i = 1, \ldots, m$ stands for the index of the ith row of the matrix, and $j = 1, \ldots, m$ stands for its jth column, defines a *linear transformation* (or

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map) T from $\mathbb{R}^m$ into itself, that associates to every vector $x = (x_1, \ldots, x_m)$ a new vector $x' = Tx$ of coordinates $x'_i = \sum_j a_{ij}x_j$, for $i = 1, \ldots, m$, or in matrix notation, $x' = Ax$. Then if $e_1, \ldots, e_m$ is the standard basis of $\mathbb{R}^m$, the vector represented by the jth column of A, i.e. $a^j = (a_{1j}, \ldots, a_{mj})$, is the image of e_j by T (or A), that is $a^j = Te_j = Ae_j$. The image $x' = Ax$ of any vector $x = (x_1, \ldots, x_m)$ is the linear combination of the vectors a^j, with weights x_j:

$$x' = A\left(\sum_j x_j e_j\right) = \sum_j x_j a^j.$$

Conversely, any linear transformation T from $\mathbb{R}^m$ into itself can be (uniquely) represented by the matrix $A = [a_{ij}]$, in the standard basis, where the jth column $a^j = (a_{1j}, \ldots, a_{mj})$ of the matrix A is the image by T of the vector e_j. It follows from these remarks that a matrix A is *invertible* if and only if the corresponding linear transformation T is onto (i.e. the image of $\mathbb{R}^m$ by T is $\mathbb{R}^m$ itself) or equivalently, if and only if the m vectors a^j are *linearly independent* (i.e. $\sum_j \alpha_j a^j = 0$ implies $\alpha_j = 0$ for all j).

A given linear transformation T of $\mathbb{R}^m$ into itself has different equivalent matrix representations, according to which basis of $\mathbb{R}^m$ is chosen. Consider a *new basis* of $\mathbb{R}^m$, i.e. a collection of m vectors $\bar{e}_1, \ldots, \bar{e}_m$, that are linearly independent. Let $(p_{1j}, \ldots, p_{mj})$ be the coordinates of $\bar{e}_j$ in the standard basis, and P stand for the matrix of which the jth columns is $\bar{e}_j$, i.e. $P = [p_{ij}]$. We know from the previous paragraph that P has an inverse P^{-1}. A vector of $\mathbb{R}^m$ of which the coordinates in the old (standard) basis are $x = (x_1, \ldots, x_m)$, has coordinates $y = (y_1, \ldots, y_m)$ in the new basis. That is, this vector can be (uniquely) expressed as a linear combination of the vectors $\bar{e}_j$ of the new basis, with weights y_j, i.e. $\sum_j y_j\bar{e}_j$. The relationship between new and old coordinates is obtained from the vector equalities

$$\sum_i x_i e_i = \sum_j y_j \bar{e}_j = \sum_j y_j \left(\sum_i p_{ij} e_i\right),$$

which imply $x_i = \sum_j p_{ij} y_j$ for all i, or in matrix notation, $x = Py$, $y = P^{-1}x$.

A given linear transformation T is represented, in matrix notation, by the map $x \to x' = Ax$ in the standard basis, and by $y \to y' = Bx$ in the new basis. Analytically, the matrix B is obtained from A by making the change of variables $x = Py$, which yields $B = P^{-1}AP$. Here again, the j-column of B represents the coordinates, in the new basis, of the image of $\bar{e}_j$ by T.

A linear transformation T of $\mathbb{R}^m$ into itself may thus be given a convenient matrix representation, by choosing an appropriate basis. The remainder of this section is devoted to such a matrix representation, the *real canonical (or Jordan) form* of T.

We look first at the circumstances ensuring that T has a *block diagonal* matrix representation. Let $E_1, \ldots, E_r$ be a collection of (linear) subspaces of $\mathbb{R}^m$, i.e. each E_h is a subset of $\mathbb{R}^m$ that is closed under the operations of addition and scalar multiplication: if x, y are vectors of E_h and α a real number, then $x + y$ and αx belong also to E_h. Assume that any vector x of $\mathbb{R}^m$ has a unique representation of the form $x = x_1 + \cdots + x_r$, in which x_h is in E_h for each h. We say then that $\mathbb{R}^m$ is the *direct sum* of the linear subspaces. Assume further that each subspace E_h is *invariant* by T, i.e. if x belongs to E_h, then Tx is also in E_h. Choose now a basis for each E_h, and take the union of the basis elements of the E_h to obtain a basis for $\mathbb{R}^m$. In that basis, T has the block diagonal form

$$B = \mathrm{diag}\{B_1, \ldots, B_r\} = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{bmatrix}.$$

This means that the matrix B_h are put together corner-to-corner diagonally as indicated, all other entries in B being zero (we adopt the convention that the blank entries in a matrix are zeros). Each matrix B_h represents in fact the restriction T_h of T to the invariant suspace E_h.

Conversely, assume that $\mathbb{R}^m$ has a basis in which T has a matrix representation of the above block diagonal form. Let E_h be the linear subspace spanned by the vectors of the basis, the images of which are the columns of the matrix B associated to the submatrix B_h. Then E_h is invariant by T, and $\mathbb{R}^m$ is the direct sum of the E_h. To sum up,