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Eight degrees of separation

Paolo Pin *

Dipartimento di Economia Politica, Università degli Studi di Siena, Piazza San Francesco 7, 53100 Siena, Italy

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ABSTRACT

We present a network formation game whose equilibria are undirected networks. Every connected couple contributes to the aggregate payoff by a fixed quantity, and the outcome is split between players according to the Myerson value allocation rule. This setup shows a wide multiplicity of non-empty equilibria, all of them connected. We show that the efficient equilibria of the game are either the empty network, or a network whose diameter does not exceed a threshold of 8 (i.e. there are no two nodes with a distance greater than 8).

1. Introduction

Network models are a good approximation of many social and economic environments, where a node is an economic agent and a link between two nodes is the possibility for both of them to communicate, exchange goods or collaborate. Applications range from the most intuitive networks of human relations, such as friendship and cooperation, to diplomatic, trade or research agreements between countries or firms. These kinds of relations might be concisely described as environments where agents optimize the gain from connections and intermediations, with the trade-off of a cost for maintaining their links (see Jackson (2006) or Vega-Redondo (2007) for a survey of all the applications in the literature).

The statistical properties of social networks have been tested in the last decade, the random graph model of Erdös and Rènyi (1960) being the benchmark model. The present work will consider the *small world* property.[1] We define the distance between two nodes as the length of the shortest path between them (infinity if they are not connected), and the diameter of the network as the maximum distance over all possible couples. A growing network will obey the small world effect if, as the number of its nodes increases, its diameter grows less than the logarithm of the number of nodes (which is the asymptotic limit in a random graph). The property was defined *small world* by Watts (1999); it dates however back to popular folklore (e.g. every US citizen is at five handshakes from the President), dramas,[2] and to a famous experiment conducted by sociologist Milgram (1967). The small world property does not appear only in social networks but also in natural and human-made physical structures.

Models of network formation have been proposed since the late 90s in two separate research fields. Game-theoretic models of network formation, from the pioneering paper of Jackson and Wolinski (1996), address a classical economics problem. A network can be thought of as the result of all its nodes solving the following optimization problem: on the one hand they seek a central role in the network, which would maximize the profit from connections (not only the direct ones) and (in some models) the probability of being necessary for other couples to connect; on the other hand they try to limit the cost of direct connections. Almost all the subsequent models in the literature have either hypotheses under which the equilibrium of this game is unique, or they focus on a range of parameters that guarantees uniqueness.[3] In the original

* Tel.: +39 0577 232692; fax: +39 0577 232661.
E-mail address: pin3@unisi.it.

1 Newman (2003) and Jackson and Rogers (2007) illustrate other peculiar properties of social networks.

2 The play *Six degrees of separation*, by Guare (1990), also became a Hollywood movie.

3 In this literature the main point is to highlight the incompatibility between stability (equilibria) and efficiency. See Jackson (2003) for a survey. As we will see, this is not an issue in our model, where we actually analyze efficient equilibria.

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Connection model of Jackson and Wolinski (1996), but also in more recent works such as in Goyal and Vega Redondo (2007), this equilibrium has the shape of a star, so that, even if all the players of the game are *ex ante* homogeneous, one of them, in equilibrium, will be connected to all the $N-1$ others, which are connected only to her. The star network trivially satisfies the statistical properties of social networks, e.g. as far as the small world property is concerned, its diameter is bounded by 2.

Another approach, starting from Albert and Barabási (1999), proposes stochastic processes of growing networks, where at every instant in discrete time a new node enters and attaches itself to the previous nodes, according to probabilistic rules. The resulting architectures have an expected topology that, depending on the specifications, satisfies some of the statistical properties of social networks. In this sense the best similarity to real networks, so far, has been reached by Jackson and Rogers (2007).

The present work describes a game-theoretic network formation model, where both the resulting network and the payoffs depend deterministically on the strategy profiles of the agents. This model is not much different from previous ones, its variables being only the size N of the network and the constant cost of forming links (which is scaled so that the payoff is normalized); however, it shows a wide multiplicity of equilibria. We will focus our attention on the efficient ones and show that their diameter is (non-trivially) always bounded by 8, so that they satisfy, as N grows, the small world property.

Section 2 describes the model, in the framework of a game, with the notion of equilibrium known as *pairwise stability*. Section 3 shows the intermediate and final propositions, while Section 4 concludes. We leave most of the mathematics to the Appendix, which is devoted to rigorous proofs.

2. The model

We imagine a finite number N of economic agents (individuals, firms, etc.) playing a simultaneous undirected network formation game. Our strategy profiles are the original ones of Jackson and Wolinski (1996). The possible action of any agent i is to make or not make a proposal of link formation to every one of the $N-1$ other agents. In this way a strategy is an array of intended links. The resulting network will be the one in which a link between agent i and agent j is present if and only if both agents made a proposal to the other to form that link.

Our agents are intended as traders or collaborators who need connections (directly or indirectly) to extract a surplus from their joint work, as we will formalize below. However, they also bear a fixed bilateral cost $c > 0$ for every link they have, so that the aggregate cost of all the network is $2 \cdot c \cdot L(G)$, where $L(G)$ is the total number of links in the network G.

In order to characterize a network formation game we need to define some basic notions, a value function, an allocation rule and a concept of stability.

2.1. Preliminary definitions

We start by giving some formal definitions for finite networks.[4] Let us consider a set N of nodes, where $N \geq 3$ will also indicate the number of nodes. A network G is a set of links between the nodes, formally $G \subseteq N \times N$. G is *undirected* and *irreflexive* if any link is an undirected couple of distinct elements from N. A link will be any such couple $g_{i,j} \equiv \{i,j\} \in G$. We call *graph architecture* the class of equivalence that can be obtained with permutations of the elements of N. Subgraph of G will be a synonym of subset.

Given a graph G on N, ambiguity can be maintained, when the context allows it, between a subset $S \subseteq N$ and the resulting subgraph $S \equiv \{\{i,j\} \in G : i \in S, j \in S\}$.

We call $l(i)$ the number of links involving i (the degree of i) and $L(G)$ the total number of links in G (so that $\sum_{i \in N} l(i) = 2 \cdot L(G)$). If $S \subseteq N$ we indicate by $L(S)$ the total number of links in G between elements of S.

Every G on N defines a topology on it. A *path* $X_{i,j} \subseteq G$ between i and j is an ordered set of agents $\{i, i_2, \ldots, i_n, j\} \subseteq N$ such that $\{g_{i,i_2}, g_{i_2,i_3}, \ldots, g_{i_n,j}\} \subseteq G$. We will write X instead of $X_{i,j}$ when the context allows it. $|X|-1$ is the length of the path, where $|X|$ is the typical notation for the cardinality of the set X. If $X_{i,j} \subseteq G$ exists we say that i and j are *connected* in G (we will write $i \sim_G j$, or even $i \sim j$). Consider a subset of the nodes $S \subset N$, we will write $i \sim_S j$ if there is path $X_{i,j} \subseteq G$ such that all the nodes in the path (even i and j) are members of S.

A *queue* is a graph consisting of a single path. A path $X_{i,i}$ from i to itself is a *cycle* (whose length is always greater than 1 in irreflexive graphs). A *circle* is a graph consisting of a single cycle.

The *distance* between i and j in G is $d_G(i,j) \equiv \min\{|X_{i,j}|-1\}$ if a path between i and j exists (we will write simply $d(i,j)$ when possible), otherwise $d_G(i,j) \equiv \infty$. The *diameter* of a graph is $D_G \equiv \max\{d_G(i,j) : i,j \in G\}$.

The definition of a *component* is straightforward; it is the set of all the nodes connected to a certain node i: $\Gamma_G(i) \equiv \{(i,j) : i \sim_G j\} \subseteq G$. G is *connected* if $D_G < \infty$, which means that for any $i \in N$, $\Gamma_G(i) = G$ (i.e. there is only one component). When a graph is connected the distance makes our topology a metric.[5]

[4] We try to integrate the original notation of Jackson and Wolinski (1996), as it has been enriched in more recent papers (such as Jackson (2005), or Goyal and Vega Redondo (2007)) into a mathematical setup that is necessary in our proofs and clarifies some of the possible sources of ambiguity.

[5] d_G is sometimes referred to as geodesic distance (i.e. the shortest path allowed) but here we do not have any other distance to distinguish it from.