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Scale and conformal invariance in field theory: a physical counterexample

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Abstract

In this Letter, we illustrate how the two-dimensional theory of elasticity provides a physical example of field theory displaying scale but not conformal invariance.

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1. Introduction

In the quantum field theory literature, scale invariance is often assumed to imply conformal invariance, provided the theory is local. Furthermore, both invariances are usually considered equivalent to the tracelessness of the stress–energy tensor. These widely held convictions, sustained by the difficulty of finding counterexamples, are actually incorrect.

Coleman and Jackiw [1] clarified this issue in the case of four space–time dimensions, showing that conformal invariance is not in general guaranteed by the presence of scale invariance. A systematic analysis of the problem for arbitrary dimensionality D was then performed by Polchinski [2], who achieved the same conclusion for any $D \neq 2$. In the particular case $D = 2$, however, Polchinski proved that scale invariance implies conformal invariance under broad conditions. In the following, we will focus on this interesting dimensionality, providing a physical example in which the implication does not hold.

Let us now summarize the observations presented in Ref. [2]. Given a symmetric and conserved stress–energy tensor $T_{\mu\nu}(x)$, the property of scale invariance can be equivalently formulated in terms of its

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trace as

$$T_\mu{}^\mu(x) = -\partial_\mu K^\mu(x), \tag{1}$$

where $K^\mu(x)$ is some local operator. Conformal invariance further requires the existence of another local operator $L(x)$ such that

$$K_\mu(x) = -\partial_\mu L(x) \ \Rightarrow\ T_\mu{}^\mu(x) = \partial_\mu\partial^\mu L(x). \tag{2}$$

The above property is then equivalent to the tracelessness of the stress–energy tensor, because one can define the 'improved' tensor

$$\Theta_{\mu\nu}(x) = T_{\mu\nu}(x) + \partial_\mu\partial_\nu L(x) - g_{\mu\nu}\partial_\rho\partial^\rho L(x), \tag{3}$$

which is both conserved and traceless. As properly emphasized in Ref. [2], most of the physically relevant theories display both scale and conformal invariance because they do not have any non-trivial candidate for K_μ. We will see in the following how this is the crucial ingredient in our counterexample.

Besides these general remarks, Polchinski also refined an argument by Zamolodchikov [3], demonstrating that scale invariance implies conformal invariance in $D = 2$. The proof consists of defining another kind of 'improved' stress–energy tensor $\Theta'_{\mu\nu}(x)$, whose trace is shown to have a vanishing two-point function:

$$\left\langle \Theta'_\mu{}^\mu(x)\Theta'_\sigma{}^\sigma(0)\right\rangle = 0. \tag{4}$$

The sufficient condition for constructing $\Theta'_{\mu\nu}(x)$ is a discrete spectrum of scaling dimensions, and, together with the assumption of reflection positivity, (4) implies the vanishing of the trace $\Theta'_\mu{}^\mu$ itself. Actually, under the above hypotheses the two 'improved' tensors $\Theta_{\mu\nu}(x)$ and $\Theta'_{\mu\nu}(x)$ coincide.

2. The model

Let us now introduce a physical example in which scale invariance does not imply conformal invariance. This is the theory of elasticity [4] in two dimensions, defined by the Euclidean action

$$\mathcal{S} = \int d^2x\,\mathcal{L} = \frac{1}{2}\int d^2x\left\{2g u_{\mu\nu}u^{\mu\nu} + k\left(u_\sigma{}^\sigma\right)^2\right\}, \tag{5}$$

where $u_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu + \partial_\nu u_\mu)$ is the so-called strain tensor, built with the 'displacement fields' u_μ. Greek indices run over $1, 2$ and we use the summation convention. The coefficients g and $k+g$ represent, respectively, the shear modulus and the bulk modulus of the described material.

The action (5) is invariant under translations, rotations and dilatations, provided the fields u_μ transform under rotations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ as vectors

$$u'_\mu(x') = \Lambda_\mu{}^\nu u_\nu(x), \tag{6}$$

while no change is required for fields under dilatations. The canonical stress–energy tensor

$$T^{c}_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu u_\sigma)}\partial_\nu u_\sigma - g_{\mu\nu}\mathcal{L} \tag{7}$$

associated to (5) is traceless but not symmetric. However, a symmetric and conserved tensor $T_{\mu\nu}$ can be conventionally constructed via the Belinfante prescription:

$$T_{\mu\nu} = T^{c}_{\mu\nu} + \partial^\rho B_{\rho\mu\nu}, \tag{8}$$

where

$$B_{\rho\mu\nu} = \frac{i}{2}\left\{\frac{\partial\mathcal{L}}{\partial(\partial^\rho u_\sigma)}S_{\nu\mu}u_\sigma + \frac{\partial\mathcal{L}}{\partial(\partial^\mu u_\sigma)}S_{\rho\nu}u_\sigma + \frac{\partial\mathcal{L}}{\partial(\partial^\nu u_\sigma)}S_{\rho\mu}u_\sigma\right\} = -B_{\mu\rho\nu}. \tag{9}$$

$S_{\mu\nu}$ is an antisymmetric tensor, taking values in the representations of the Lorentz group, which expresses the variation of the field multiplet $\phi = \{u_\mu\}$ under infinitesimal rotations $x'^\mu \simeq x^\mu + \omega^\mu_\nu x^\nu$:

$$\phi'(x') \simeq \left(I - \frac{i}{2}\omega_{\rho\nu}S^{\rho\nu}\right)\phi(x). \tag{10}$$

In our case the fields transform according to the vector representation (6), and the only non-vanishing Euclidean components of $S_{\mu\nu}$ act as

$$S_{12}u_1 = -S_{21}u_1 = iu_2,$$
$$S_{12}u_2 = -S_{21}u_2 = -iu_1.$$

It follows from (8) that the trace of the stress–energy tensor can be cast in the form (1)

$$T_\mu{}^\mu = -\partial^\mu K_\mu \quad \text{with} \quad K_\mu = -B_{\mu\rho}{}^\rho, \tag{11}$$

in agreement with the scale invariance of the theory. In order to investigate whether the additional property (2), equivalent to conformal invariance, is also