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Sigma model Lagrangian for the Heisenberg group

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Abstract

We study the Lagrangian for a sigma model based on the non-compact Heisenberg group. A unique feature of this model—unlike the case for compact Lie groups—is that the Lagrangian has to be regulated since the trace over the Heisenberg group is otherwise divergent. The resulting theory is a real Lagrangian with a quartic interaction term. In particular, in $D = 2$ space–time dimensions, after a few non-trivial transformations, the Lagrangian is shown to be equivalent, at the classical level, to a complex cubic Lagrangian. A one-loop computation confirms that the quartic and cubic Lagrangians are equivalent at the quantum level as well.

The complex Lagrangian is known to be classically equivalent to the $SU(2)$ sigma model, with the equivalence breaking down at the quantum level. An explanation of this well-known results emerges from the properties of the Heisenberg sigma model.

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1. Introduction

Sigma models in two-dimensional space–time have a long history in theoretical physics [1]. They are ubiquitous in particle physics, with many applications and extensions in quantum field theory and string theory. These sigma models are usually based on compact Lie groups.

In this Letter, we construct a sigma model based on the non-compact Heisenberg group. The present study is motivated by the more complex case studied in [2] which deals with a supersymmetric Yang–Mills theory having a local infinite-dimensional Kac–Moody group as its gauge group. The need for regulating the Lagrangian of the theory was essential in obtaining local Kac–Moody gauge symmetry. The Heisenberg alge-

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bra is an infinite-dimensional non-compact subalgebra of the well-known Kac–Moody algebra [3], and the Heisenberg sigma model is the simplest theory having the new features that emerge from constructing quantum field theories based on infinite-dimensional Lie algebras. Coupled to the fact that there is a central extension associated with the Heisenberg algebra, the sigma model obtained here will be different from the sigma model based on a compact Lie group. This is the main motivation for studying the Lagrangian obtained in this Letter.

Given that the group element of the Heisenberg group is infinite-dimensional, the usual procedure of obtaining a Lagrangian by tracing over a representation of the group yields a divergent result. We regulate the trace to obtain a finite Lagrangian, which turns out to be a real quartic Lagrangian $\mathcal{L}_4$. Since no restriction was imposed in obtaining the Lagrangian, the result is valid for arbitrary space–time dimensions D.

Furthermore, one can show that in two dimensions, after some straightforward calculations involving functional integrals, the quartic Lagrangian is equivalent to a complex cubic Lagrangian $\mathcal{L}_3$. Interestingly enough, the relationship of the two Lagrangians is *not* that of a usual duality transformation since the mapping does not induce an inversion of the coupling constant. The cubic Lagrangian $\mathcal{L}_3$ in turn is known to be *classically* equivalent to the sigma model based on the $SU(2)$ group [4] which has spontaneous particle production [5]. Furthermore, it is also known that this equivalence breaks down on quantizing the two theories [5]. The causes of this breakdown was expounded in [6,7] and explored in [8,9]. Since we are dealing with a model based on an infinite-dimensional Lie group, our work should provide an understanding of the quantum inequivalence from a different perspective.

It is well known that a pair of quantum theories are equivalent if the corresponding correlation functions of both theories are equal. Since our cubic Lagrangian is obtained from that of the quartic Lagrangian by performing an exact Gaussian integration and by constant field rescalings which do not affect the path integral measure, the quantum theory of the two Lagrangians are equivalent.

To verify the quantum equivalence of the cubic and quartic theories, we compute the one-loop beta function for the two theories. We employ the background field method to study the renormalizability of the theory up to the one-loop correction. A similar, but much more complex, calculation was carried out in [10] to study the renormalizability of a $U(1)$ gauge field with Kac–Moody gauge symmetry. We calculate the β-functions for both the quartic and cubic Lagrangian realizations of the theory, and show that to one-loop they are identical. In so doing we also verify the one-loop renormalizability of two (apparently dissimilar) bosonic theories.

2. The Heisenberg sigma model Lagrangian

Consider the (non-compact) Heisenberg algebra $[x, p] = ik$, where k is the central extension. In terms of the creation and destruction operators, it is given by $[a, a^\dagger] = k$. Since we would like to construct a sigma model based on the Heisenberg group, thus let us start from the finite group elements of the Heisenberg algebra. By the usual exponential mapping, we can write such an element as

$$\Omega = \exp\left[i\phi + i\omega a + i\omega^* a^\dagger\right]. \tag{1}$$

Note that the field (group coordinate) ϕ is a real variable, whereas the field ω is a complex variable. The field ϕ has to be introduced due to the existence of the central extension of the Heisenberg algebra.

The simplest non-linear sigma model Lagrangian based on a space–time dependence of the group coordinates ϕ and ω is defined by

$$\mathcal{L} = \mathrm{Tr}\left[\partial_\mu \Omega^\dagger \partial_\mu \Omega\right]. \tag{2}$$

However, this approach fails since the trace over the non-compact operators a, $a^\dagger$ diverges, yielding $\mathcal{L} = \infty$. A similar situation was encountered in defining the supersymmetric gauge fields with the infinite-dimensional Kac–Moody symmetry [2].

To successfully obtain a finite Lagrangian, one must regularize the trace $\mathrm{Tr}[\cdots]$ over the infinite-dimensional operators. There is a wide variety of regulators which one can choose, and we expect from the principle of universality that a whole range of regulators would lead to the same renormalizable theory [2].[1] We make the natural choice for the regulator of

[1] Note that another approach of regularizing this Lagrangian may be developed from the works of Wigner and Moyal [12].