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Invariants of lepton mass matrices and CP and T violation in neutrino oscillations

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Abstract

CP and T asymmetries in neutrino oscillations, in vacuum as well as in matter, are expressed in terms of invariant functions of lepton mass matrices.

1. Introduction

Four decades have passed since the unexpected discovery [1] of CP violation, originally seen in two decay modes of K_L. For about 25 years the superweak ansatz by Wolfenstein [2] accounted for all observed effects, including CP violation in the semileptonic decay modes of K_L, and provided a simple and intuitive explanation for why CP violation was not seen elsewhere in particle physics. Nowadays the standard framework for understanding CP violation is the electroweak model [3] (hereafter referred to as the standard model) and the Kobayashi–Maskawa scheme [4] within it. Indeed the remarkable experimental determination of the quantity ϵ'/ϵ in K_L decays and the discovery of CP violation in the decays of B-mesons, at SLAC and KEK are in agreement with the predictions of the standard model with three families. Nonetheless, understanding CP violation still remains a great challenge, within a broad area in physics. One faces deep questions such as what is the source of CP violation that goes into generating the baryon asymmetry of the universe and why is the theta parameter of QCD so small that it has not been seen. There are also some perhaps simpler questions, one being: is there CP violation in the leptonic sector? The minimal standard model gives a no as the answer to the latter question because it assumes that the neutrinos are massless. But nowadays there is evidence for neutrino oscillations, a phenomenon that requires massive neutrinos [5]. Therefore, the minimal standard model needs to be modified, but we do not know how. In this Letter, I shall address the issue of CP violation in neutrino oscillations using the method of invariant functions of mass

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matrices [6,7]. In this Letter, first a short introduction to this method is presented in the next section followed by application to neutrino oscillations, in vacuum as well as in matter, in the following sections.

2. Invariant functions of mass matrices

Consider first the quark sector of the standard model with three families. The identity of the quarks is encoded in the three-by-three quark mass matrices M_u and M_d, for the up-type and down-type quarks, respectively. However, these mass matrices are basis dependent. Given any pair M_u, M_d one can obtain other pairs through unitary rotations, as will be described below, without affecting the physics. The measurable quantities must be basis independent and, therefore, they are "invariant functions" under such rotations. These functions were introduced in [6] and studied in detail in [7]. Actually, what enters, in the standard model, is the pair

$$S_u \equiv M_u M_u^\dagger, \qquad S_d \equiv M_d M_d^\dagger. \tag{1}$$

For simplicity, we shall refer to these quantities as mass matrices. This should cause no confusion because the underlying mass matrices, M_u and M_d, do not enter in what follows.

An invariant function $f(S_u, S_d)$ is a function that does not change under the transformation

$$S_u \to X S_u X^\dagger, \qquad S_d \to X S_d X^\dagger, \tag{2}$$

where X is an arbitrary three-by-three unitary matrix. Evidently, the traces of powers of the above quantities, $\mathrm{tr}(S_u^k S_d^l)$, are such invariant functions. Note that the corresponding determinants are not independent invariant functions because any determinant can be expressed as a function of traces. For a detailed discussion of these invariants see [7].

When dealing with CP violation in the standard model with three families, a central role is played by the commutator of the quark mass matrices, $[S_u, S_d]$ (see [6,8]). The determinant of this commutator is an invariant function of mass matrices given by [6,8]

$$\det[S_u, S_d] = 2iJ v(S_u) v(S_d), \tag{3}$$

where J is the CP-invariant of the quark mixing matrix V,

$$\mathrm{Im}\big(V_{\alpha j} V_{\beta k} V^\star_{\alpha k} V^\star_{\beta j}\big) = J \sum_{\gamma, l} \epsilon_{\alpha\beta\gamma}\epsilon_{jkl}. \tag{4}$$

J is equal [9] to twice the area of any of the six by now well-known unitarity triangles. The quantities $v(S_u)$ and $v(S_d)$ are Vandermonde determinants as follows. Denoting the three eigenvalues of S_u by x_i, $x_1 = m_u^2$, $x_2 = m_c^2$, $x_3 = m_t^2$, we have

$$\begin{aligned} v(S_u) &= \sum_{i,j,k} \epsilon_{ijk} x_j x_k^2 \\ &= \big(m_u^2 - m_c^2\big)\big(m_c^2 - m_t^2\big)\big(m_t^2 - m_u^2\big) \end{aligned} \tag{5}$$

and similarly $v(S_d) = (m_d^2 - m_s^2)(m_s^2 - m_b^2)(m_b^2 - m_d^2)$.

In the standard model with three families, the non-vanishing of the above determinant gives the if and only if condition for CP violation in the quark sector. In fact it manifestly unifies the 14 conditions needed, such as the condition that no two quarks with the same charge are allowed to be degenerate if CP is to be violated, or the conditions that none of the mixing angles nor the phase angle is allowed to assume its maximal or minimal value. In other words, the commutator automatically keeps track of these requirements and is thus useful for checking whether a specific model violates CP or not (for a pedagogical discussion see [10]).

The determinant in Eq. (3) appears naturally in computations involving CP violation when all the six quarks enter on equal footing. Examples are the renormalization of the θ-parameter of QCD by the electroweak interactions and the calculation of the baryon asymmetry of the universe in the standard model. The determinant, in a truncated form, enters in many more computations such as when computing the electric dipole moment of a quark, say the down quark. Since in such a computation, the down quark appears in the external legs, it is tacitly assumed that we know the identity of this quark, i.e., $m_d \neq m_s$ and $m_d \neq m_b$. Therefore, the factors $(m_d^2 - m_s^2)$ and $(m_b^2 - m_d^2)$ will be missing but all the other factors will be present.