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Selfduality of $d = 2$ reduction of gravity coupled to a σ-model

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Abstract

Dimensional reduction in two dimensions of gravity in higher dimension, or more generally of $d = 3$ gravity coupled to a σ-model on a symmetric space, is known to possess an infinite number of symmetries. We show that such a bidimensional model can be embedded in a covariant way into a σ-model on an infinite symmetric space, built on the semidirect product of an affine group by the Witt group. The finite theory is the solution of a covariant selfduality constraint on the infinite model. It has therefore the symmetries of the infinite symmetric space. (We give explicit transformations of the gauge algebra.) The usual physical fields are recovered in a triangular gauge, in which the equations take the form of the usual linear systems which exhibit the integrable structure of the models. Moreover, we derive the constraint equation for the conformal factor, which is associated to the central term of the affine group involved.

1. Introduction

Dimensional reduction of gravity in dimension $d = 4$ to $d = 2$ enlarges the group of symmetries to an infinite group [1], which has been related to the integrable structure of the theory [2,3]. The symmetry group appeared to be an affine Kac–Moody group [4–6]. Furthermore, it was shown that this structure was shared by a full class of theories, such as supergravities, which in dimension three reduce (for the bosonic sector) to a symmetric space σ-model coupled to gravity [4,5,7].

In addition to the scalars which describe a $\mathfrak{G}/\mathfrak{H}$ σ-model, the (bosonic) degrees of freedom of these models consist of a dilaton ρ and the conformal factor $\lambda = e^{\sigma}$ of the metric. The equations of motion are, in conformal gauge,

$$\mathrm{d} * \mathrm{d}\rho = 0, \tag{1}$$

$$\nabla(\rho * P) = 0 \tag{2}$$

(P and ∇ are precisely defined below) with in addition a first order constraint on the conformal factor,

$$\partial_\pm \rho \partial_\pm \hat{\sigma} = \frac{1}{2}\rho\langle P_\pm, P_\pm\rangle. \tag{3}$$

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where $\hat{\sigma} = \sigma - \frac{1}{2}\ln(\partial_+\rho\partial_-\rho)$. We choose a Lorentzian spacetime for exposition; it is not hard to adapt the results for a Euclidean one.

In [8,9], the symmetry of such a theory was enlarged to the full semidirect product of the Witt group and an affine Kac–Moody group. The fields are infinitely dualised and live in an infinite coset

$$\mathcal{M} = \frac{\mathcal{W} \ltimes \mathfrak{G}^\infty}{\mathcal{K} \ltimes \mathfrak{H}^\infty}. \tag{4}$$

In this formalism, the equations of motion come from linear systems which are imposed as constraints in a triangular gauge.

Here, following [10], which deals with the flat space σ-model, we restore the infinite $\mathcal{K} \ltimes \mathfrak{H}^\infty$ gauge-invariance: we define the finite, constrained model through a covariant selfduality equation on the infinite tower of fields. Previously known Lax pairs are recovered as consequences of this constraint when we go into the triangular gauge. This gives a formulation of the $d = 2$ theory very analogous to the oxidised versions ($d \geqslant 3$) [11–13].

We also derive the constraint for the conformal factor from the selfduality constraint. Whereas for other fields the duality involves a Hodge dualisation, it is worth noticing that this is not the case for the conformal factor, associated to the central term of the group.

In Section 2, we describe the infinite symmetric space $\mathcal{M}$ and the algebraic structures involved. In Section 3, we define the duality operator and we show that a selfduality constraint can be imposed and is covariant with respect to the infinite gauge algebra transformations. Finally, we fix the gauge in Section 4 to recover the physical content of the theory, and we derive the Lax pair equations associated to the dynamical fields of the model, together with the conformal factor constraint.

2. Infinite σ-model structure

Following [8], we consider fields in an infinite-dimensional symmetric space

$$\mathcal{M} = \frac{\mathcal{W} \ltimes \mathfrak{G}^\infty}{\mathcal{K} \ltimes \mathfrak{H}^\infty}. \tag{5}$$

$\mathcal{W}$ is the "group" of diffeomorphisms of the real line, $\mathfrak{G}^\infty$ is the affine extension $\mathfrak{G}^{(1)}$ of the simple group $\mathfrak{G}$ and $\mathcal{K} \ltimes \mathfrak{H}^\infty$ is the subgroup of fixed points of $\mathcal{W} \ltimes \mathfrak{G}^\infty$ under some involution.

Explicitly, $\mathfrak{G}^\infty$ is the set of pairs $(g(t), a)$ where $g(t)$ is a map from $\mathbb{R}_+^\times$ to $\mathfrak{G}$ and a is a positive real number. The group law is

$$\big(g_1(t), a_1\big)\big(g_2(t), a_2\big) = \big(g_1(t)g_2(t), a_1 a_2 e^{\Omega(g_1,g_2)}\big), \tag{6}$$

where Ω is a group 2-cocycle (see [6]). The Lie algebra is the affine Kac–Moody algebra $\mathfrak{g}^\infty$ with analytic functions $b(t)$ with values in $\mathfrak{g}$ and in addition a central charge c; commutation relations are

$$\big[b_1(t), b_2(t)\big] = \big[b_1(t), b_2(t)\big]_{\mathfrak{g}} + \omega(b_1, b_2)c, \tag{7}$$

where ω is a 2-cocycle of the loop algebra (see [6,14]). Here, it is defined as

$$\omega(b_1, b_2) = \frac{1}{2}\oint_{\mathcal{C}_1} \mathrm{d}t \,\big\langle \partial_t b_1(t), b_2(t)\big\rangle + \frac{1}{2}\oint_{\mathcal{C}_2} \mathrm{d}t \,\big\langle \partial_t b_1(t), b_2(t)\big\rangle, \tag{8}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two contours exchanged and reversed under $t \to 1/t$, avoiding singularities [6].

$\mathcal{W}$ is, at least formally, the group $\mathrm{Diff}^+(\mathbb{R}_+^\times)$ of analytic diffeomorphisms of the real line preserving the orientation. We see it as Laurent series $f(t)$ with the law group given by composition:

$$(f_1 \circ f_2)(t) = f_1\big(f_2(t)\big). \tag{9}$$

Its Lie algebra is the real Witt algebra with generators $L_n = t^{n+1}\partial_t$ ($n \in \mathbb{Z}$) and commutation relations

$$[L_m, L_n] = (n - m)L_{m+n}. \tag{10}$$

The semidirect product $\mathcal{W} \ltimes \mathfrak{G}^\infty$ is given by triples $(f, g, a) \in \mathcal{W} \times \mathfrak{G}^\infty$ with product law

$$(f_1, g_1, a_1)(f_2, g_2, a_2) = \big(f_1 \circ f_2, (g_1 \circ f_2)g_2, a_1 a_2 e^{\Omega(g_1 \circ f_2, g_2)}\big). \tag{11}$$

The subgroup $\mathcal{K} \ltimes \mathfrak{H}^\infty$ consists of fixed points under an involution $\tau_\ltimes$, which is given by an involution on $\mathcal{W}$

$$\tau_{\mathcal{W}} : f(t) \to \frac{1}{f(1/t)} \tag{12}$$