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Canonical quantization of Galilean covariant field theories

E.S. Santos [a], M. de Montigny [a,b,*], F.C. Khanna [a,c]

[a] Theoretical Physics Institute, University of Alberta, Edmonton, Alta., Canada T6G 2J1
[b] Faculté Saint-Jean, University of Alberta, Edmonton, Alta., Canada T6C 4G9
[c] TRIUMF, 4004, Wesbrook Mall, Vancouver, BC, Canada V6T 2A3

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Abstract

The Galilean-invariant field theories are quantized by using the canonical method and the five-dimensional Lorentz-like covariant expressions of non-relativistic field equations. This method is motivated by the fact that the extended Galilei group in $3 + 1$ dimensions is a subgroup of the inhomogeneous Lorentz group in $4 + 1$ dimensions. First, we consider complex scalar fields, where the Schrödinger field follows from a reduction of the Klein–Gordon equation in the extended space. The underlying discrete symmetries are discussed, and we calculate the scattering cross-sections for the Coulomb interaction and for the self-interacting term $\lambda\Phi^4$. Then, we turn to the Dirac equation, which, upon dimensional reduction, leads to the Lévy-Leblond equations. Like its relativistic analogue, the model allows for the existence of antiparticles. Scattering amplitudes and cross-sections are calculated for the Coulomb interaction, the electron–electron and the electron–positron scattering. These examples show that the so-called 'non-relativistic' approximations, obtained in low-velocity limits, must be treated with great care to be Galilei-invariant. The non-relativistic Proca field is discussed briefly.

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* Corresponding author. Fax: +1 7804658760.
E-mail addresses: esantos@phys.ualberta.ca (E.S. Santos), montigny@phys.ualberta.ca (M. de Montigny), khanna@phys.ualberta.ca (F.C. Khanna).

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1. Introduction

This article is a continuation of a recent article where Galilei-invariant theories of scalar fields have been quantized [1]. It lies within the scope of a program which exploits a five-dimensional covariant formulation of Galilean covariance to understand non-relativistic field theories [2–4]. In these articles, we have applied an extended space-time approach such as devised in [5,6]. There exist similar procedures, discussed in [7,8]. This has been utilized recently to study fluid dynamics [9,10]. The appearance of this approach in physics has been ubiquitous. It probably dates back to nearly 35 years ago, when Susskind investigated the perturbative behaviour of quantum electrodynamics in the limit of high-velocity processes, more specifically, the use of infinite-momentum frame for strong interactions [11]. He was using what is referred to nowadays as the light-front formalism [12]. As far as we are concerned, a remarkable achievement in these papers was to notice the appearance of the $2+1$ Galilean group related to the motions transverse to the direction of the infinite momentum.

A wealth of non-relativistic phenomena, particularly in condensed matter physics, low-energy nuclear physics, and many-body theory [13], are likely to benefit from any such convenient tool. Indeed, although it is usually understated, Galilean invariance is crucial for the applications of the methods of quantum field theory to many low-temperature systems such as superfluids, superconductors, and Bose–Einstein condensation. A first advantage of the extended space-time formalism is that Galilean covariance is manifest throughout the calculations, in the same way that Lorentz covariance is manifest in relativistic theories. Therefore, the various procedures involved are carried out in a way quite similar to the relativistic ones. Another, rather technical, advantage is that projective representations may be avoided when one is willing to pay the price of working in $4+1$ space-time. Finally, since the Poincaré group in $4+1$ dimensions clearly contains as a Lie subgroup the usual Poincaré group of $3+1$ dimensions, in addition to the Galilei group of $3+1$ space-time, then we may derive not only Galilean invariant results, but also Lorentz invariant results simply by using a different reduction [6,7]. This most desirable feature has been utilized in the investigation of fluid dynamics [9,10].

The formalism begins with a extended space $\mathcal{G}_{(4,1)}$, which is actually a Minkowski space in $4+1$ dimensions. Group theoretically speaking, we simply exploit the fact that the (centrally extended) Galilei group in $3+1$ space-time is a subgroup of the Poincaré group in $4+1$ dimensions. Hereafter, we will not repeat the different physical motivations for the enlarged manifold, since it can be found in the references mentioned earlier. Let us just state that we work with *Galilean five-vectors* $(\mathbf{X}, X^4, X^5)$, which transform under Galilean boosts as:

$$\mathbf{X}' = \mathbf{X} - \mathbf{V}X^4,$$

$$X^{4'} = X^4,$$

$$X^{5'} = X^5 - \mathbf{V}\cdot\mathbf{X} + \frac{1}{2}\mathbf{V}^2 X^4,$$