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The construction of spinors in geometric algebra

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Abstract

The relationship between spinors and Clifford (or geometric) algebra has long been studied, but little consistency may be found between the various approaches. However, when spinors are defined to be elements of the even subalgebra of some real geometric algebra, the gap among algebraic, geometric, and physical methods is closed. Spinors are developed in any number of dimensions from a discussion of spin groups, followed by the specific cases of U(1), SU(2), and SL(2, $\mathbb{C}$) spinors. The physical observables in Schrödinger–Pauli theory and Dirac theory are found, and the relationship between Dirac, Lorentz, Weyl, and Majorana spinors is made explicit. The use of a real geometric algebra, as opposed to one defined over the complex numbers, provides a simpler construction and advantages of conceptual and theoretical clarity not available in other approaches.

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1. Introduction

Spinors are used in a wide range of fields, from the quantum physics of fermions and general relativity, to fairly abstract areas of algebra and geometry. Independent

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of the particular application, the defining characteristic of spinors is their behavior under rotations: for a given angle θ that a vector or tensorial object rotates, a spinor rotates by $\theta/2$, and hence takes two full rotations to return to its original configuration. The *spin groups*, which are universal coverings of the rotation groups, govern this behavior, and are frequently defined in the language of geometric (Clifford) algebras [1,2].

In this paper, we follow the geometric algebra approach to its logical conclusion, and define spinors as arbitrary elements of the *even subalgebra* of a real geometric algebra; since the spin group is made up of normalized even multivectors, the action of a rotation maps spinor space onto itself. The identification of spinors with real even multivectors in geometric algebra was first made by David Hestenes [3–5]; we extend his approach using group theory and insights provided by algebraic spinor methods (see, e.g., [6]).

Many modern mathematical treatments ([7,8], for example) begin by defining a *complex* geometric algebra, in which the representation of the spin group lives. Spinors are then written as members of left minimal ideals of the Clifford algebra. Although this method is closely related to our treatment, the geometrical interpretation is muddied by the presence of the imaginary unit *i*. In addition, spinors in the left-ideal approach lie in the full geometric algebra, rather than its even subset, and the specific left minimal ideal is dependent on the full algebra. By contrast, if spinors are assigned to the same algebra that defines the spin group, they may be embedded in algebras of higher dimension. Left ideals do continue to play a role, however, and are used in relating Dirac spinors to two-component (Lorentz) spinors in spacetime.

The advantage of defining spinors over the field of real numbers, however, lies in the fact that every multivector in a real geometric algebra has a geometrical interpretation (however complicated). In fact, the introduction of imaginary units in spinor theory arose out of matrix representations of spin groups, but when the same group is represented in a real geometric algebra, the introduction of complex numbers is superfluous. (The relationship between matrix and geometric algebra approaches to group theory is obtained by representing a geometric algebra as a matrix algebra; see Appendix A.) The complex (Hermitian) structure of spinors is found by specifying a "spin-axis" in the space under consideration, so that it depends not only on the dimensionality of the group, but also on an orientation for the space. Although we will not discuss topics in topology in this paper, the existence of such a complex structure seems to be intimately related to the existence of global spin structure in a manifold (see [9,10]). Since defining spinors in a real algebra is extensible to any dimension, all the results of standard spinor theory can be carried over.

Section 2 introduces spin groups for any dimension and signature in the language of geometric algebra, defining rotations in terms of multivector objects. The next sections deal with specific spin groups, and their associated spinors: Section 3 treats two-dimensional Euclidean and anti-Euclidean spinors, Section 4 is concerned with Pauli spinors in three-dimensions, while spacetime spinors are the subject of Sections 5 and 6. In each case, we will show that the complex structure deemed necessary to treat spinors arises naturally from the real geometric algebra; the connection to