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Connecting the spatial structure of periodic orbits and invariant manifolds in hyperbolic area-preserving systems

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Abstract

This Letter discusses the equivalence between the Bowen measure associated with the set $Per(n)$ of periodic points of period n of hyperbolic area-preserving maps of a smooth manifold, and the measure associated with the intersections between stable and unstable manifolds of hyperbolic points. In typical cases of physical interest (i.e., nonuniformly hyperbolic systems) these measures are found to be highly nonuniform (multifractal).

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Periodic orbits represent the fundamental invariant structure around which hyperbolic (chaotic) dynamics is organized [1]. Given a map Φ, which is chaotic within an invariant region $\mathcal{C}$, $\Phi(\mathcal{C}) = \mathcal{C}$, the set $Per(n)$ of periodic orbits of period $n = 1, 2, \ldots$ is defined by $Per(n) = \{\mathbf{x} \mid \Phi^n(\mathbf{x}) = \mathbf{x}\}$. Moreover, the union of all the periodic points is dense in $\mathcal{C}$, and all the periodic orbits falling within the closure of $\mathcal{C}$ are unstable. To give an example, in dissipative systems with a nontrivial attractor, it has been showed that the set $Per(n)$ of periodic points of period n at increasing n can be used as the support for a sequence of weighted atomic measures that converge to the invariant measure of the attractor as n diverges to infinity [2]. In this case, the weight at any given periodic point $\mathbf{x}$ is related to the average stretch factor along the periodic orbit passing through $\mathbf{x}$. The connection between the atomic measures supported on $Per(n)$ and the ergodic measure supported on the attractor, together with the observation that the structure of $Per(n)$ is also related to other physically relevant properties such as shadowing [3], motivated a considerable effort oriented towards the definition of efficient numerical algorithms for computing periodic points of relatively high period [4]. A review of the approaches and perspectives based on stabilization transformations to detect unstable peri-

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odic orbits can be found in [5], while the extension of these methods in the case of continuous-time dynamical systems is addressed in [6].

In this Letter, we show that the spatial structure of $Per(n)$ and the related measure-theoretical properties of the set of unstable periodic orbits are closely connected with fundamental geometric properties of the dynamics in conservative chaotic dynamical systems. In the case of measure-preserving systems, and specifically for volume preserving transformations, the physically meaningful measure supported on $Per(n)$ is not that associated with a stretch-factor weighting function, but is represented by the measure that assigns to each periodic point of this set a uniform weight. Specifically, the resulting sequence of measures defined for periodic points of maximum period n has been proved to converge, as $n \to \infty$, to an invariant measure, which is referred to as the Bowen measure μ_{Bow}, after Bowen [7] (see Eq. (3)).

In massively chaotic systems (i.e., nonuniformly hyperbolic systems according to Pesin's definition [8]), there exists yet another fundamental invariant geometric structure, namely, that associated with stable and unstable manifolds densely filling the chaotic region. For any periodic point x^* of prime period m, $\Phi^m(\mathbf{x}^*) = \mathbf{x}^*$, the stable and unstable manifolds $\mathcal{W}^s_{\mathbf{x}^*}$, $\mathcal{W}^u_{\mathbf{x}^*}$ of $\mathbf{x}^*$ are defined as:

$$\mathcal{W}^s_{\mathbf{x}^*} = \left\{\mathbf{x} \,\middle|\, d\left(\mathbf{x}^*, \Phi^{nm}(\mathbf{x})\right) \to 0 \text{ for } n \to \infty\right\},$$
$$\mathcal{W}^u_{\mathbf{x}^*} = \left\{\mathbf{x} \,\middle|\, d\left(\mathbf{x}^*, \Phi^{-nm}(\mathbf{x})\right) \to 0 \text{ for } n \to \infty\right\},$$

where $d(\cdot,\cdot)$ is a distance function. These manifolds are invariant, i.e., $\Phi(\mathcal{W}^s_{\mathbf{x}^*}) = \mathcal{W}^s_{\mathbf{x}^*}$ and $\Phi(\mathcal{W}^u_{\mathbf{x}^*}) = \mathcal{W}^u_{\mathbf{x}^*}$. The role of the stable and unstable manifolds in the occurrence of chaotic behavior has been thoroughly analyzed [9]. Homoclinic and heteroclinic intersections of invariant manifolds are a signature of chaos and play a significant role in the phenomenology of scattering problems [10] as in the properties of Hamiltonian systems [11]. Starting from the knowledge of the unstable manifold of a hyperbolic fixed point, it is indeed possible to determine the stability region in the phase space, at least for simple systems [12].

In two-dimensional area-preserving dynamical systems, invariant manifolds define the asymptotic structure attained by material lines (i.e., passive interfaces) as well as a hierarchy of spatial structures (lobes) useful to model transport in chaotic systems [11,13]. The measure-theoretical quantification of the spatial distribution of stable and unstable invariant manifolds can be achieved through the concept of w-measures [14] (for more details see Eq. (5) and the related discussion).

There is a sort of "correspondence principle" between the characterization of hyperbolic dynamical systems based on these two approaches (periodic orbits vs invariant manifolds), and this Letter is devoted to the quantitative understanding of this duality. Specifically, the aim of this Letter is: (i) to analyze the spatial structure of the Bowen measure associated with $Per(n)$ for two-dimensional area-preserving uniformly and nonuniformly hyperbolic systems, and (ii) to relate this measure to the w-measures associated with the spatial distribution of the unstable/stable manifolds.

On a global scale, the first indication that such relation should exist is represented by the equality between the exponent yielding the asymptotic growth of periodic points and the line stretching exponent, which, in the two-dimensional case, coincides with the topological entropy of the transformation [15].

As a prototypical system for our investigation, we consider the area-preserving homeomorphism of the 2d-torus $\mathcal{H}: T^2 \to T^2$ given by

$$\mathcal{H}(\mathbf{x}) = \mathcal{H}(x, y) = \left(x + f(y), y + x + f(y)\right) \bmod 1,$$

where $f(\xi): [0,1] \to [0,1]$ is the tent map ($f(\xi) = 2\xi$ for $\xi \leqslant 1/2$ and $f(\xi) = 2 - 2\xi$ for $\xi > 1/2$), (x, y) being the coordinates on the unit square interval $I^2 = [0,1) \times [0,1)$, equipped with periodic boundary conditions ("mod 1"), which represents a global projection chart for the two-torus T^2.

The map $\mathcal{H}$ is area-preserving, globally continuous, yet not C^1-smooth. In fact, the torus meridians $y = 0$ and $y = 1/2$ are singularity lines for the differential $\mathcal{H}^*(\mathbf{x})$ of $\mathcal{H}$ ($\mathcal{H}^*(\mathbf{x})$ is the Jacobian matrix of the transformation). The differential $\mathcal{H}^*(\mathbf{x})$ is piecewise constant and equal to

$$H_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \text{for } 0 \leqslant x < 1,\ 0 < y < 1/2,$$
$$H_0 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \quad \text{for } 0 \leqslant x < 1,\ 1/2 < y < 1. \qquad (1)$$