Available online at www.sciencedirect.com
SCIENCE DIRECT®
PHYSICS LETTERS A
Physics Letters A 346 (2005) 99–110
www.elsevier.com/locate/pla

# Existence of localized solutions in the parametrically driven and damped DNLS equation in high-dimensional lattices ☆

Yu Feng [a,b], Wen-Xin Qin [a,*], Zhigang Zheng [c]

[a] *Department of Mathematics, Suzhou University, Suzhou 215006, PR China*
[b] *Department of Applied Mathematics, Nanjing Audit University, Nanjing 210029, PR China*
[c] *Department of Physics, Beijing Normal University, Beijing 100875, PR China*

Received 12 April 2005; received in revised form 12 June 2005; accepted 15 July 2005

Available online 3 August 2005

Communicated by C.R. Doering

**Abstract**

Instead of using the homoclinic orbit approach, which was commonly taken when studying the localized solutions of the discrete non-linear Schrödinger (DNLS) equation in one-dimensional lattices, we apply the continuation theorem to investigate the existence, stability, and spatial complexity of the localized solutions, including bright breathers, dark breathers, and anti-phase breathers, of the parametrically driven and damped DNLS equation in high-dimensional lattices. In particular, we prove that the sufficient condition that the driving strength exceeds the damping constant is necessary for the system with weak coupling to possess localized solutions.


*PACS:* 02.30.Hq; 02.30.Sa; 05.45.-a; 63.20.Pw

## 1. Introduction

The discrete non-linear Schrödinger (DNLS) equation, which describes a particularly simple model for a lattice of coupled anharmonic oscillators, attracts much attention in recent years [10,17,20]. Most effects are devoted to the research of the standard DNLS equation [10]. In this Letter, we consider DNLS equation with damping and driving force. In fact, the parametrically driven and damped DNLS equation can be regarded as the discretization of the corresponding continuum NLS equation. Moreover, when the application of the continuum NLS equation is

☆ Supported by the Special Funds for Major State Basic Research Projects, National Natural Science Foundation of China and Natural Science Foundation of Jiangsu Province.
* Corresponding author.
*E-mail address:* wxqin@public1.sz.js.cn (W.-X. Qin).

0375-9601/$ – see front matter 
doi:10.1016/j.physleta.2005.07.070

considered, the discrete structure has to be taken into account and we must consider the discretized system rather than the continuum equation [18].

In order to describe soliton dynamics in non-linear media, the NLS equation including damping and parametric driving terms, $i\psi_t + \Delta\psi + 2|\psi|^2\psi = h\psi^* e^{i\Omega t} - i\gamma\psi$, has been extensively studied [3–5,28]. If the driving force $h$ exceeds the damping strength $\gamma$, then soliton solutions $\psi_+$ and $\psi_-$ exist, which can be found exactly [4]. $\psi_-$ exists and is unstable for $\gamma < h < \sqrt{(\Omega/2)^2 + \gamma^2}$, while the stability of $\psi_+$ was studied [4] by reducing the stability problem for the dissipative NLS to the stability problem for the dissipative-free case. $\psi_+$ is unstable for $h > \sqrt{(\Omega/2)^2 + \gamma^2}$ with respect to the continuous spectrum. The stability diagram was depicted in [3,4].

Meanwhile, taking into account the discrete structure of the system and numerical study of the non-linear PDE, one should consider the discrete system, i.e., the DNLS equation. In fact, much work has been devoted to the localized solutions of the DNLS equation, see [1,6,17,18,23,24]. By localized solutions, or discrete breathers (DBs), we mean that they are spatially localized and time-periodic. For localized solutions in high-dimensional lattices, see [11,12,15,22]. Recent work on the localized solutions of the DNLS equation without damping and driving in high-dimensional lattices can be found in [7,21].

In this Letter, we consider in high-dimensional lattices the parametrically driven and damped DNLS equation

$$i\dot{\psi}_s + 2|\psi_s|^2\psi_s + \alpha \sum_{r\in N_s} (\psi_r - \psi_s) = h\psi_s^* e^{i\Omega t} - i\gamma\psi_s, \tag{1.1}$$

where the real parameters $h > 0$ and $\gamma > 0$ determine the strength of driving and damping respectively, $s$ belongs to an index set $Z^d$, the $d$-dimensional integer lattice with $d \geqslant 1$, $\alpha \in R$ is the coupling coefficient, $N_s$ denotes the set of nodes coupled to site $s$, the star in the right hand side indicates complex conjugation. For one-dimensional case, i.e., $d = 1$, with the nearest neighbor coupling, i.e., $N_s = \{s+1, s-1\}$, Hennig [18] studied numerically the rich behavior of the localized solutions by taking a dynamical systems approach: Substituting the ansatz $\psi_n(t) = \phi_n e^{i\omega t}$ (we denote the site index by $n$ for one-dimensional case) with $\omega = \Omega/2$ yields an algebraic equation of the complex amplitudes $\phi_n$, which can be converted to a four-dimensional volume-preserving map. Thus a bright breather solution is given by a homoclinic orbit associated with a hyperbolic fixed point at the origin. A transversal homoclinic orbit is determined by computing the stable and unstable manifolds of the hyperbolic fixed point. The stability of the breather solution is analyzed numerically on the basis of the Floquet theory.

Unfortunately, the approach taken in [18] cannot be generalized to high-dimensional cases, as pointed out by Flach [14]. Roughly speaking, there are two methods studying the existence of DBs in lattice systems, as introduced in [13]. One is called anti-integrability developed by MacKay and Aubry [25,26], the other homoclinic orbit approach proposed by Flach [14] for the systems with homogeneous potentials. In fact, the problem of DBs for the DNLS equation can also be discussed by the homoclinic orbit approach [1,6,17,18,23]. In particular, the one-dimensional case of (1.1) with the nearest neighbor coupling was studied by such approach. We remark here that in one-dimensional case, the site index $n$ plays the role of time when converting to a low-dimensional map. When the space dimension is greater than one, the algebraic equation of the spatial amplitudes derived from (1.1) cannot be transformed to some finite-dimensional map.

The second point we want to emphasize is that the analysis of the stability of DBs was carried out [18] numerically. In fact, as we will show in Section 3, the stability can be analyzed rigorously due to the detailed discussion of the single oscillator.

In order to apply perturbation methods such as averaging procedure, the driving and damping parameters $h$ and $\gamma$ were restricted to be small [18]. We hope to remove such restriction.

The existence of DBs has been extensively studied since the pioneering work of MacKay and Aubry [26]. However, the non-existence problem is rarely investigated so far, to our knowledge. We show the non-existence of DBs for system (1.1) with small coupling if $h < \gamma$. Therefore, the sufficient condition $h > \gamma$ is necessary for system (1.1) with weak coupling to possess localized solutions.

The Letter is organized as follows. In Section 2, we summarize the results of the periodic solutions of the single oscillator in order to apply the continuation theorem. We then investigate in Section 3 the existence, stability, and