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Singularity of free-surface curvature in convergent flow: Cusp or corner?

Yulii D. Shikhmurzaev

School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK

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Abstract

The nature of singularities that arise in the mathematical modeling of free-surface flows and the ways of their analysis and regularization aimed at removing the physically unacceptable features is one of fundamental issues in theoretical fluid dynamics. The present work considers the type of the free-surface curvature singularity emerging in the steady two-dimensional convergent flow of a Newtonian fluid near a free boundary. The unphysical singularities in the flow field, unavoidable in the conventional model, are removed by describing this flow as a particular case of the interface formation/disappearance process in the framework of an earlier developed macroscopic theory of such processes which is applied without any ad hoc alterations. The near-field asymptotic analysis of the problem shows that at finite capillary numbers the singularity of the free-surface curvature is always a sharp corner, not a cusp.

1. Introduction

A theoretical possibility that a free surface between two immiscible fluids can have a singularity of curvature due to the fluids' motion was first investigated by Richardson [1], who considered the deformation of a liquid–vacuum interface with a constant surface tension in the Stokes flow. His elegant analysis based on the technique of conformal mapping produced exact analytical results showing that, if one presumes a priori that there is a singularity of curvature at a finite capillary number, then this singularity can only be a cusp pointing into the fluid with the stream function in its vicinity, to leading order in r as $r \to 0$, given by

$$\psi(r,\theta) = \frac{\sigma_e}{2\pi\mu} r \log r \sin\theta, \tag{1}$$

where r and θ are polar coordinates in the plane of flow with the axis of symmetry $\theta = 0$ pointing into the fluid, μ is the fluid's viscosity and σ_e is the surface tension at the free surface. It should be noted, however, that according to this solution, the rate of energy

E-mail address: yulii@for.mat.bham.ac.uk (Y.D. Shikhmurzaev).

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dissipation in the flow is nonintegrable and hence the total dissipation of energy in the vicinity of the cusp is infinite.

Experiments reported by Joseph et al. [2] and repeated in a slightly different flow configuration by Jeong and Moffatt [3] demonstrated that a singularity of curvature can exist in reality and indeed it *looks* like a cusp. More importantly, experiments [3] have shown that the appearance of the singularity is associated with a qualitative change in the flow kinematics: the fluid particles belonging to the free surface ('marked' by the powder sprinkled on it) no longer stay there at all times; instead, they are swept through the singularity into the interior of the fluid. Jeong and Moffatt investigated the phenomenon theoretically, again in the framework of the standard model but without presuming a priori that there is a singularity of free-surface curvature. Their exact analytical solution, also obtained by conformal mapping, shows that for the convergent Stokes flow generated by a dipole placed underneath a free surface the radius of curvature scaled with the characteristic lengthscale of the fluid motion, R/d, decreases exponentially with the capillary number Ca ($= \mu U/\sigma_e$; U is the characteristic velocity of the fluid):

$$R/d \sim \frac{256}{3}\exp(-32\pi Ca). \tag{2}$$

Then, as the capillary number increases towards that of the apparent cusping observed in experiments (and the associated qualitative change in the flow kinematics), the radius of curvature of the free surface, which in the model is, by definition, a macroscopic quantity, goes into the region of molecular and then submolecular length scales. In other words, the solution falls outside the limits of applicability of the model in the framework of which it was obtained. This outcome of the modeling is important per se, irrespective of experiments, and the latter merely confirm that the problem is a real one. (In some experiments, air entrainment is reported to occur before the formation of the singularity of curvature [4], whereas in others it is specifically emphasized that the singularity is observed without any evidence of air entrainment [2,3] which eventually develops only after a further increase in the flow rate [2]. In any case, the presence of air is a physical factor *additional* to the essential hydrodynamics of the convergent flow and hence its influence can be manipulated. For example, one can delay air entrainment thus allowing the singularity to form by placing a low-volatile fluid in a low-pressure chamber, similar to how air entrainment is postponed in experiments on dynamic wetting [5].)

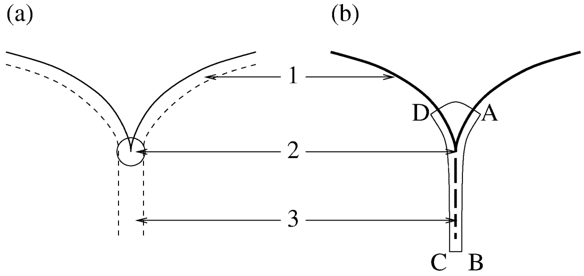

Fig. 1. Schematic illustration of the physical picture of the flow near a singularity of free-surface curvature. In the continuum approximation, the interfacial layers 1(a) are modeled as sharp interfaces 1(b), the transition region 2(a) is seen as a contact line 2(b), and the surface-tension-relaxation tail 3(a) becomes a gradually disappearing internal interface 3(b). For the total force acting on the control volume ABCD, with the boundaries AB and CD lying, physically, just outside the range of intermolecular forces, to be zero, the surface tensions acting on AD can only be balanced by shear stress acting on AB and CD. Then, to balance this stress at every point of the internal interface there must be the surface-tension gradient along it.

Thus, the convergent flow considered in the framework of the conventional model presents a paradox: there is either the solution (1) with a singularity at a finite capillary number and the associated infinite dissipation of energy, or the solution leading to (2) and hence an emerging singularity as the model falls outside its limits of applicability. It should be emphasized that, given that both of the above solutions are exact, the paradox cannot be attributed to (over)simplifications made in obtaining them; it is inherent in the model itself.

The shortcomings of the standard model were removed in [6] where the line singularity of curvature was described as a 'contact line' formed at the intersection of the free surface and an 'internal interface' (Fig. 1). The existence of this interface is suggested by the flow kinematics observed experimentally: as the free surface is swept through the contact line into the interior of the fluid [3], its surface properties will have to vanish, i.e. to relax to new equilibrium values, and hence there will be a surface-tension-relaxation tail (i.e. an 'internal interface') stretching from the contact line into the bulk. It was shown that this approach