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Symmetry reductions of a $2+1$ Lax pair

P.G. Estévez [a,*], M.L. Gandarias [b], J. Prada [c]

[a] *Area de Fisica Teorica, Universidad de Salamanca, Spain*
[b] *Departamento de Matemáticas, Universidad de Cádiz, Spain*
[c] *Departamento de Matemáticas, Universidad de Salamanca, Spain*

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Abstract

In this Letter we present the reductions arising from the classical Lie symmetries of a Lax pair in $2+1$ dimensions. We obtain several interesting reductions and prove that, by analyzing not only a PDE but also its associated linear problem, it is possible to obtain the reduction of the PDE together with the reduced Lax pair. Specially relevant is the fact that the spectral parameter in $1+1$ dimensions appears as a natural consequence of the reduction itself and is related to the symmetry of the $2+1$ eigenfunction.

Keywords: Exactly solvable and integrable systems

1. Introduction

The identification of the Lie symmetries of a given partial differential equation (PDE) is an instrument of primary importance in order to solve such an equation [1]. A standard method for finding solutions of PDEs is that of reduction using Lie symmetries: each Lie symmetry allows a reduction of the PDE to a new equation with the number of independent variables reduced by one [2,3]. In a certain way this procedure gives rise to the ARS conjecture [4] which establishes that a PDE is integrable in the Painlevé sense [5] if all its reductions pass the Painlevé test [6]. This means that solutions of a PDE can be achieved by solving its reductions to ordinary differential equations (ODE). Classical [1] and nonclassical [2,3] Lie symmetries are the usual way for identifying the reductions.

* Corresponding author.
E-mail address: pilar@usal.es (P.G. Estévez).

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Nevertheless, let us recall that there are some methods for solving PDEs that are more effective in $2+1$ than in $1+1$ dimensions [7]. A good example of this is the following equation in $2+1$ dimensions

$$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = h_{yz}, \tag{1}$$

which some of us have studied in a recent paper [7] proving that the singular manifold method [6] is a very effective method for solving the equation. It is quite straightforward to determine the associated linear problem through this method. In fact, for (1) we obtained the following Lax pair:

$$-\psi_y + \psi_{xxx} + 3h_x\psi_x + \frac{3}{2}h_{xx}\psi = 0, \qquad 2h_z\psi_{xz} - h_{xz}\psi_z + 2h_z^2\psi = 0. \tag{2}$$

Notice that there are a lot of papers related with the reduction of $2+1$ equations through the classical Lie method but the application of this method to the Lax pair is much less frequent [8]. Nevertheless we consider that for integrable equations, it is of primary importance to determine the reduction, not only of the equation, but of the Lax pair. Actually the reduction process should introduce a spectral parameter that is absolutely essential in $1+1$ dimensions. Therefore, our plan in this Letter is to a certain extent exactly the opposite of the usual approach: we try to obtain $1+1$ spectral problems arising from a $2+1$ Lax pair. Once we have solved the problem in $2+1$ dimensions in [7], in the sense that we have determined its Lax pair, we shall identify the classical symmetries of the Lax pair [8]. This is done in Section 2. In Section 3 we use these symmetries to obtain a reduced Lax pair in $1+1$ dimensions whose compatibility condition should be a reduction of (1). Actually, there are five possible reductions. Two of them yield linear equations that can be easily integrated. The other three reductions yield $1+1$ spectral problems that include, as particular cases, well-known equations such as the modified Korteveg–de Vries, Drinfel'd–Sokolov or Ermakov–Pinney equations. It is interesting to note that each of these reductions yields respectively *two, three and fourth order spectral problems which exhibit a spectral parameter as a natural output of the Lie method*. We close with a section of conclusions.

2. Classical symmetries

In order to apply the classical Lie method to the system of PDEs (2) with three independent variables and two fields, we consider the one-parameter Lie group of infinitesimal transformations in x, y, z, h, ψ, given by:

$$\begin{aligned} &x' = x + \varepsilon\xi_1(x, y, z, h, \psi) + O(\varepsilon^2), \qquad y' = y + \varepsilon\xi_2(x, y, z, h, \psi) + O(\varepsilon^2),\\ &z' = z + \varepsilon\xi_3(x, y, z, h, \psi) + O(\varepsilon^2), \qquad h' = h + \varepsilon\phi_1(x, y, z, h, \psi) + O(\varepsilon^2),\\ &\psi' = \psi + \varepsilon\phi_2(x, y, z, h, \psi) + O(\varepsilon^2), \end{aligned} \tag{3}$$

where ε is the group parameter. It is therefore necessary that this one transformation leaves the set of solutions of (2) invariant. This yields an overdetermined linear system of equations for the infinitesimals $\xi_1(x, y, z, h, \psi)$, $\xi_2(x, y, z, h, \psi)$, $\xi_3(x, y, z, h, \psi)$, $\phi_1(x, y, z, h, \psi)$ and $\phi_2(x, y, z, h, \psi)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form:

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \xi_3\frac{\partial}{\partial z} + \phi_1\frac{\partial}{\partial h} + \phi_2\frac{\partial}{\partial \psi}. \tag{4}$$

By applying the classical method [1] to the system of PDEs (2), we obtain the following system of determining equations (we have used MACSYMA and MAPLE independently to handle the calculations):

$$0 = \frac{\partial\xi_1}{\partial z} = \frac{\partial\xi_1}{\partial h} = \frac{\partial\xi_1}{\partial\psi} = \frac{\partial\xi_2}{\partial x} = \frac{\partial\xi_2}{\partial z} = \frac{\partial\xi_2}{\partial h} = \frac{\partial\xi_2}{\partial\psi} = \frac{\partial\xi_3}{\partial x} = \frac{\partial\xi_3}{\partial y} = \frac{\partial\xi_3}{\partial h} = \frac{\partial\xi_3}{\partial\psi},$$