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PHYSICS LETTERS A
Physics Letters A 343 (2005) 85–89
www.elsevier.com/locate/pla

New exact traveling wave solutions for the two-dimensional KdV–Burgers and Boussinesq equations

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Received 25 February 2004; accepted 1 June 2005

Available online 13 June 2005

Communicated by A.R. Bishop

Abstract

By using the solutions of elliptic equation, a direct method is described to construct the exact traveling wave solutions for two-dimensional KdV–Burgers and Boussinesq equations. More Jacobi elliptic function solutions are obtained. These solutions can be degenerative to hyperbolic function solutions and trigonometric function solutions when the modulus m of Jacobi elliptic function is driven to limit 1 and 0. The results include solitary wave solutions, periodic wave solutions and shock wave solutions. Many new results are presented.

Keywords: Jacobi elliptic function solutions; Non-linear partial differential equations; Periodic wave solution

1. Introduction

It is well known that the non-linear partial differential equations (NPDEs) are widely used to describe many important phenomena in physics, biology, chemistry, etc. To date, significant progress in construction of the traveling wave solutions of non-linear partial differential equations is made. There are many direct methods for obtaining these traveling solution, namely, tanh function method [1–3], sech function method [4,5] and Jacobi elliptic function method [6–8]. Many research works have been concentrated on the various extensions and applications of the methods to simplify the routine of calculations. Recently, a method called the mapping method [9] has been developed to obtain different types of traveling wave solutions. The crucial point of this method is to find different types of traveling wave solutions, if they exist, simultaneously to the equation in question. In addition, using this method one can avoid many tedious and repetitive calculations.

Therefore, in this Letter, the extended mapping method is proposed to get more solutions for two-dimensional KdV–Burgers and Boussinesq equations

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.06.004

simultaneously. By this method, we can get the degenerative and non-degenerative type of solutions.

2. Applications

2.1. KdV–Burgers equation

Consider the two-dimensional KdV–Burgers equation

$$(u_t + uu_x + \alpha u_{xxx})_x + \gamma u_{yy} = 0, \tag{1}$$

where α and γ are real parameters and subscripts means differentiation. This equation is considered as the governing equation for two-dimensional waves propagation in fluid filled elastic or viscoelastic medium in which the effects of dispersive and non-linearity are present. We first consider the following formal traveling wave solutions

$$u = u(\zeta), \quad \zeta = x + y + \lambda t, \tag{2}$$

where λ is a constant. Substituting Eq. (2) into Eq. (1) yields the ordinary non-linear differential equation

$$(\lambda u_\zeta + uu_\zeta + \alpha u_{\zeta\zeta\zeta})_\zeta + \gamma u_{\zeta\zeta} = 0. \tag{3}$$

In the second step we expand $u(\zeta)$ as follows

$$u(\zeta) = \sum_{i=0}^{n} a_i f^i(\zeta), \tag{4}$$

where a_i are constants to be determined and $n = 2$ is determined by balancing the linear term of the highest order derivative with non-linear term, while f satisfying the following elliptic equation

$$f''(\zeta) = pf(\zeta) + qf^3(\zeta) + sf^5(\zeta). \tag{5}$$

Upon integrating (5), yields

$$f'(\zeta) = \sqrt{pf^2(\zeta) + \frac{1}{2}qf^4(\zeta) + \frac{1}{3}sf^6(\zeta) + r}. \tag{6}$$

Here the prime means derivatives with respect to ζ. Substituting (4) and (5) into Eq. (3) and collecting coefficients of power of f^i,

$$f^i\sqrt{pf^2(\zeta) + \frac{1}{2}qf^4(\zeta) + \frac{1}{3}sf^6(\zeta) + r},$$

then setting each coefficient to zero, we can deduce the following set of over determined of algebraic equations with respect to a_i, λ, p, q, s and r

$$\begin{aligned}
&6r\big(a_1^2 + 2a_2(a_0 + \lambda + 4p\alpha + \gamma)\big) = 0,\\
&6a_1\big(a_0 p + \lambda p + 6r(a_2 + q\alpha) + p(p\alpha + \gamma)\big) = 0,\\
&12\big(a_1^2 p + a_2\big(2a_0 p + 2\lambda p + 3r(a_2 + 6q\alpha)\\
&\quad + 2p(4p\alpha + \gamma)\big)\big) = 0,\\
&6a_1\big(9a_2 p + (a_0 + \lambda)q + 10pq\alpha + 20rs\alpha + q\gamma\big) = 0,\\
&3\big(3a_1^2 q + 2a_2\big(8a_2 p + 3(a_0 + \lambda)q + 60pq\alpha + 80rs\alpha\\
&\quad + 3q\gamma\big)\big) = 0,\\
&2\big(3a_1\big(6a_2 q + 6q^2\alpha + s(a_0 + \lambda + 26p\alpha + \gamma)\big)\big) = 0,\\
&2\big(15a_2^2 q + 4a_1^2 s + 2a_2\big(45q^2\alpha\\
&\quad + 4s(a_0 + \lambda + 40p\alpha + \gamma)\big)\big) = 0,\\
&2s\big(15a_1(a_2 + 4q\alpha)\big) = 0,\\
&2s\big(12a_2(a_2 + 20q\alpha)\big) = 0,\\
&70s^2\alpha a_1 = 0,\\
&256s^2 a_2 = 0.
\end{aligned} \tag{7}$$

Solving the whole set of Eq. (7), we obtain

$$a_0 = -\lambda - \gamma - 4\alpha p, \qquad a_1 = 0,$$
$$a_2 = -6\alpha q. \tag{8}$$

With the knowledge of the above parameters, we can write the exact solution of Eq. (3) in the form

$$u(\zeta) = -(\lambda + \gamma + 4\alpha p) - 6\alpha q f^2(\zeta), \tag{9}$$

where $p \neq 0$, $q \neq 0$ and f is the solution of

$$d\zeta = \int_0^f \frac{df'}{\sqrt{pf'^2 + \frac{1}{2}qf'^4 + \frac{1}{3}sf'^6 + r}}. \tag{10}$$

As we have mentioned above the advantage of this method is to construct different types of traveling wave solutions simultaneously. According to different choices of p, q, s and r, five cases are presented here.

Case 1. $p = -(1 + m^2)$, $q = 2m^2$, $r = 1$, $s = 0$.

With this choice, Eq. (3) has the periodic wave solution

$$\begin{aligned}u(\zeta) = {}&-\big(\lambda + \gamma - 4\alpha\big(1 + m^2\big)\big)\\ &- 12\alpha m^2\,\mathrm{sn}^2(\zeta, m).\end{aligned} \tag{11}$$