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Variational formulation of the Gardner's restacking algorithm

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Abstract

The incompressibility of the phase flow of Hamiltonian wave-plasma interactions restricts the class of realizable wave-driven transformations of the particle distribution. After the interaction, the distribution remains composed of the original phase-space elements, or local densities, which are only rearranged ("restacked") by the wave. A variational formalism is developed to study the corresponding limitations on the energy and momentum transfer. A case of particular interest is a toroidal plasma immersed in a dc magnetic field. The restacking algorithm by Gardner [Phys. Fluids 6 (1963) 839] is formulated precisely. The minimum energy state for a plasma with a given current is determined.
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1. Introduction

The incompressibility of the phase flow of Hamiltonian wave-plasma interactions restricts the class of realizable wave-driven transformations of the particle distribution [1]. This restricts the energy in a plasma available for extraction [2]. In the case where the particle interactions with waves are diffusive, the energy available for extraction is further constrained by the consideration of only diffusive phase-space rearrangements [3]. This limits, for example, the amount of energy that can be extracted from α particles in a tokamak [4].

A related problem has also been addressed recently in Refs. [5,6] in connection with generating plasma current by means of an asymmetric ponderomotive barrier for thermal particles. In this case, a one-way rf barrier is set up that can reflect particles coming from one direction, while being transparent to particles coming from the other direction. The barrier must of necessity heat the particles that pass through it in order to conserve the phase space density. This means that the current can be generated by these barriers in plasma, but only at the price of some energy expended.

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In general, all these limitations can be attributed to the existence of what can be called the “plasma ground state” for a given one-particle distribution f_1. By ground state, we mean such a distribution of particles f_2, which minimizes the total plasma energy on the manifold of all Hamiltonian transformations $f_1 \to f_2$. As reported in the pioneering paper by Gardner [1], the ground state plasma energy $W_{\min}$ is generally nonzero, which can be explained as follows. Suppose that a bounded plasma particles having the initial phase-space distribution f_1 are introduced into an electromagnetic field for a limited time, which eventually results in bringing the plasma to some final state f_2. Imagine that we partition the plasma phase space into small cells of equal volume $\Delta\Gamma_i = \Delta\Gamma$, and to each cell attach a certain value of the distribution function $f(\Gamma_i)$. As the number of cells that have a given value of f is conserved throughout the interaction (as follows from the Liouville theorem), the distribution f_2 may not be arbitrary, but rather will represent a result of reordering (“restacking”) of the original phase-space elements $\Delta\Gamma_i$, regardless of the spatial and temporal structure of the external fields. Alternatively, this fact can be expressed as conservation of the so-called Casimir invariants, which determine the distribution of the values $f(\Gamma_i)$ (see, e.g., Ref. [7]) and whose existence is an intrinsic property of any Hamiltonian system.

The plasma ground state will correspond to the distribution f_2, such that the elements $\Delta\Gamma_i$ with larger $f(\Gamma_i)$ occupy the states with lower particle energy $\mathcal{E}$. In a bounded plasma, only a finite phase volume is allotted to the states with $\mathcal{E}$ below a given value. Hence, from incompressibility of the phase flow, it follows that after the interaction the plasma will be left with the total energy $W \geqslant W_{\min}$, where $W_{\min}$ is a nonzero quantity defined as the minimum of W over all possible ways of restacking the elements $\Delta\Gamma_i$.

While chopping phase space into discrete elements is pictorial, it is fairly artificial in case of a continuous function f_1. Hence, solving the “restacking problem” must be possible in terms of a differential formulation. The purpose of the present Letter is to derive such a formulation and apply it to a number of cases of interest, not previously considered.

The Letter is organized as follows: in Section 2, we generalize the Gardner's problem by putting it into a variational form for an abstract dynamical system. We determine the condition under which a Hamiltonian transformation of the system phase space yields the maximum or minimum of a given functional (such as the plasma energy in Ref. [1]). In the framework of this formalism, we reproduce the results given in Ref. [1] and, in Section 3, solve a similar, yet different problem of finding the minimum energy state at given plasma current. In Section 4, we apply our formalism to magnetized toroidal plasmas and derive a reduced variational principle. In Section 5, we summarize our main ideas.

2. Variational formalism

Let us first restate the Gardner's problem in its original form [1]. Suppose that a bounded plasma with the initial distribution $f(\mathbf{r}_1, \mathbf{p}_1)$ is introduced into external fields for a limited time, which eventually results in bringing the plasma to some final state $f(\mathbf{r}_2, \mathbf{p}_2)$. The particle distribution is conserved: $f(\Gamma_2) = f(\Gamma_1)$, where $\Gamma_2 \equiv (\mathbf{r}_2, \mathbf{p}_2)$ is a single-valued reversible function of $\Gamma_1 \equiv (\mathbf{r}_1, \mathbf{p}_1)$. Thus, the total energy left inside the plasma after the interaction equals

$$W = \int \mathcal{E}(\Gamma_2) f(\Gamma_1)\, d\Gamma, \tag{1}$$

where $\mathcal{E}$ is the individual particle energy, and where we made use of phase space conservation: $d\Gamma \equiv d\Gamma_1 = d\Gamma_2$, $d\Gamma_i \equiv d^3r_i\, d^3p_i$. Suppose that the particles initially occupy a nonzero phase volume. In a bounded plasma, only a finite phase volume is allotted to the states with $\mathcal{E}(\Gamma_2)$ below a given value. Hence, from incompressibility of the phase flow it follows that after the interaction the plasma will be left with the total energy $W \geqslant W_{\min}$, where

$$W_{\min} = \min_{\Gamma_1 \to \Gamma_2} \int \mathcal{E}(\Gamma_2) f(\Gamma_1)\, d\Gamma \tag{2}$$

is a nonzero quantity defined as the minimum of W over all possible Hamiltonian (canonical) phase-space transformations $(\mathbf{r}_1, \mathbf{p}_1) \to (\mathbf{r}_2, \mathbf{p}_2)$. Hence, determination of $W_{\min}$ can be considered as a variational problem of searching for the canonical transformation $\Gamma_1 \to \Gamma_2$, which minimizes the functional (1).

Treated like that, the Gardner's problem yields a natural generalization as follows. Suppose one is given a function $\phi(\Gamma_1)$ defined in a $2N$-dimensional phase

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