Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Physics Letters A 340 (2005) 361–368
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Fractional description of super and subdiffusion

Remi-Ahmad El-Nabulsi

Plasma Application Laboratory, Department of Nuclear and Energy Engineering, and Faculty of Mechanical, Energy and Production Engineering, Cheju National University, Ara-dong 1, Jeju 690-756, South Korea

Received 30 September 2004; received in revised form 1 February 2005; accepted 11 March 2005

Available online 21 April 2005

Communicated by C.R. Doering

RETRACTED

Abstract

In this Letter we discuss anomalous transport in complex dynamical systems. Regular and consistent derivation of macroscopic and transport equation for systems with super and subdiffusive behavior is provided and their properties are investigated in details. By focusing on large space–time, we derive macroscopic fractional equation from microscopic stochastic equation.

1. Introduction

In the past decades, classical Hamiltonian physics has achieved unexpected and impetuous developments which had led to an entirely new understanding of the classical simple dynamical systems, an area of physics that has been assumed to be generally well understood and concluded [1]. In the last twenty years it became evident, however, that even the simplest, completely deterministic systems may show irregular, chaotic behavior that seems not easy to treat [2]. It was believed that chaotic and random stochastic behavior exists only in systems with a large amount of degree of freedom, e.g., gas. It has now been established that such a behavior can be exhibited by systems with merely two degrees of freedom [3]. For example, the motion of a particle in 2D conservative force field typically shows chaotic behavior. Since the first indication of chaos is strictly deterministic systems, a lot of scientific studies have been carried out, in which the existence of such “deterministic chaos” has been analyzed and further verified [4]. The standard map is the simplest and most frequently occurring model in many different applications [1,6]. It has become a paradigm for the study of properties of chaotic dynamics in Hamiltonian systems [1]. A more fundamental difference between perfectly chaotic systems, like Sinai’s billiard and more realistic systems, like the standard map, is the existence of islands in the phase space, making this later a complicated mixture of small non-chaotic domains and stochastic regions [5,12]. In fact, the origin of anomalous transport has been connected to the local topological properties of the phase space domain near islands (*the boundary*

E-mail address: atomicnuclearengineering@yahoo.com (R.-A. El-Nabulsi).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.03.085

islands) and a self-similar structure of the domain. The existence of a fine structure of the island patterns in phase space in Hamiltonian systems and its properties are fundamental in the study of anomalous transport. Long range correlations effects occur from visits of orbits to boundary layers in the vicinity of islands. This phenomenon is generally referred to as the stickiness of the islands. It was shown that for small changes of the stochastic parameter strong topological changes can occur in the phase portrait of the system [1,5]. Usually the theoretical investigations of random transport use two levels of description: a *microscopic* and *macroscopic* one. The first deals with the motion law of single particles, the second refer to the motion of an ensemble of particles. Of course, both of them are connected with each other, and a description of one of the levels helps to understand the other, by use of dual point of view: *the process of random drunkard walks is equivalent to diffusion and vice versa* [6]. A noteworthy point is that real and computer experiments do not succeed to investigate both side of a phenomenon simultaneously, and analytical calculus is required. While such a double approach has been done in the most classical stochastic transport examples, it still has not been done in the case of more complex motion. Indeed, physical systems with fractional power-type law of random displacement are usually investigated in detail on a microscopic level. Very few attempts have been done in the frame of macroscopic equation. For instance, Saichev et al. have analyzed the properties of a probability density satisfying fractional generalizations of Fokker–Planck and Kolmogorov–Feller equation [7,28]. By showing that such processes are a decomposition of the fractal Brownian motion and the Levy-type processes (Levy stable laws), they obtain interesting and fundamental physical interpretations of fractional kinetic equations. However, let us recall that the most well known application of Levy stable laws in physics corresponds to the anomalous diffusion or transport associated with a Levy motion, also often called a "Levy flights" [13–16]. The mean square displacement is obviously incompatible with the classical Fokker–Planck equation. The renormalization group was used to obtain an explicit expression for the transport equation. In the case of the standard map, this later is expressed as a function of space–time scaling constants for islands in the boundary layer. Following different mathematical approaches, several authors considered generalization of the Fokker–Planck equation in order to encompass the Levy anomalous diffusion. Particular cases of the fractional Fokker–Planck equation were obtained. Tsallis [17] had introduced a non-linear Fokker–Planck equation where the scaling relation is a possible solution. This proves that this is no the only way to generalize the Fokker–Planck equation in order to respect the anomalous scaling relation. However, there have been a lot of discussions concerned the way to implant the self-similarity properties into the kinetic theory. Specifically, to describe the dynamics of the Hamiltonian chaos, there have been the kinetic integral equations based on the Montroll–Weiss kinetic equation [6], the fractional Fokker–Planck–Kolmogorov equation based on the application of the fractional calculus to the fractal space–time random walks and finally the Weierstrass self-similar random walk. The introduction of self-similar properties to the singular zones by any form in the derivation of the kinetic equation, imply the appearance of new information concerning the islands structure, their topology and their bifurcations. The existence of islands and singular zones implies also a non-uniform topology in the phase space, which could be considered as the main cause of anomalous transport (non-Gaussian). To obtain a kinetic equation describing the anomalous transport, different formal approaches on fractional calculus have been studied and discussed. The major problem of these phenomenological aspects is the absence of locality.

In this Letter, we want to show that dealing with fractional derivatives is not more complex than that with usual differentiable operators. Indeed integral terms become local after Fourier and/or Laplace transformations. We are rather interested in the derivation of a fractional equation starting from a given distribution probability. We want to show a consistent derivation of equations describing stochastic but non-diffusive spreading of ensemble particles. We restrict ourselves to the one-dimensional case. This derivation is not connected with a concrete physical system. The problem we are interested in is the determination of the dynamical aspects of physical system satisfying anomalous microscopic motion. We will introduce specific models for the super and subdiffusive cases. The first case occurs in turbulent transport and generalized statistical thermodynamics, while the second is typical of transport of charge carries in disordered