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A q-deformed nonlinear map

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Abstract

A scheme of q-deformation of nonlinear maps is introduced. As a specific example, a q-deformation procedure related to the Tsallis q-exponential function is applied to the logistic map. Compared to the canonical logistic map, the resulting family of q-logistic maps is shown to have a wider spectrum of interesting behaviours, including the co-existence of attractors—a phenomenon rare in one-dimensional maps.

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1. Introduction

Emergence of the so-called quantum group structures in certain physical problems has led to studies on several q-deformed physical systems [1]. Enthused over this, and inspired by the elements of Tsallis statistical mechanics [2], we suggest a scheme of q-deformation of nonlinear maps. We then elucidate the general features of a q-deformed logistic map related to the Tsallis q-exponential function, as a concrete illustration of the scheme of q-deformation of nonlinear maps.

Theory of quantum groups turned the attention of physicists to the rich mathematics of q-series, q-special functions, etc., with a history going back to the nineteenth century [3]. The q-deformation of any function involves essentially a modification of it such that in the limit $q \to 1$ the original function is recovered. Thus there exist several q-deformations of the same function introduced in different mathematical and physical contexts. Here, we are concerned mainly with the q-deformation of real numbers and the exponential function.

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Originally, in 1846 Heine deformed a number to a basic number as

$$[n]_q = \frac{1-q^n}{1-q}, \tag{1}$$

such that $[n]_q \to n$ when $q \to 1$. In 1904 Jackson defined a q-exponential function given by

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{2}$$

with

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \qquad [0]_q! = 1, \tag{3}$$

as the solution of the q-differential equation

$$\frac{df(x)}{d_q x} = \frac{f(x)-f(qx)}{(1-q)x} = f(x). \tag{4}$$

It is seen that $E_q(x) \to \exp(x)$ in the limit $q \to 1$ when the Jackson q-differential operator $d/d_q x$ also becomes the usual differential operator d/dx.

The mathematics of quantum groups necessitated a new deformation of number as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \tag{5}$$

which also has the required property that in the limit $q \to 1$, $[n]_q \to n$. The associated q-exponential function is given by the same equation (2) but with $[n]_q$ defined according to (5).

In the nonextensive statistical mechanics of Tsallis [2], a new q-exponential function has been introduced as given by

$$e_q^x = \big(1 + (1-q)x\big)^{1/(1-q)}, \tag{6}$$

which satisfies the nonlinear equation

$$\frac{df(x)}{dx} = \big(f(x)\big)^q, \tag{7}$$

and has the required limiting behaviour: $e_q^x \to \exp(x)$ when $q \to 1$. This e_q^x plays a central role in the nonextensive statistical mechanics by replacing $\exp(x)$ in certain domains of application; it should be noted that it is natural to define a generalized exponential function as in (6) if we consider the relation

$$e^x = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N, \tag{8}$$

and regard $1/N$ as a continuous parameter. The formalism of nonextensive statistical mechanics has found applications in a wide range of physical problems [2], including the study of nonlinear maps at the edge of chaos. Here we derive another deformation of numbers based on the Tsallis q-exponential function defined by (6), and use it to study a q-deformed logistic map as an example of the general scheme of q-deformation of nonlinear maps.

2. A q-deformation scheme for nonlinear maps

The series expansion of e_q^x has been presented in [4] as

$$e_q^x = 1 + \sum_{n=1}^{\infty} \frac{Q_{n-1} x^n}{n!}, \tag{9}$$