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Dynamics of a piecewise smooth map with singularity

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Abstract

Experiments observing the liquid surface in a vertically oscillating container have indicated that modeling the dynamics of such systems require maps that admit states at infinity. In this Letter we investigate the bifurcations in such a map. We show that though such maps in general fall in the category of piecewise smooth maps, the mechanisms of bifurcations are quite different from those in other piecewise smooth maps. We obtain the conditions of occurrence of infinite states, and show that periodic orbits containing such states are superstable. We observe period-adding cascade in this system, and obtain the scaling law of the successive periodic windows.

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1. Introduction

Recently a lot of research attention has been directed toward the dynamics of piecewise smooth maps (PWS), because they represent a large number of systems of practical interest including switching electrical circuits and impacting mechanical systems. In such systems the discrete phase space is divided into compartments within which the map is smooth, and the compartments are separated by borderlines at which the map is not differentiable. A one-dimensional piecewise smooth map has the general form

$$x_{n+1} = f(x_n) = \begin{cases} g(x_n, \mu), & \text{for } x_n < \lambda, \\ h(x_n, \mu), & \text{for } x_n > \lambda, \end{cases} \tag{1}$$

where μ is the bifurcation parameter and the compartments are separated by the borderline value λ. In such a map, there is the possibility that a fixed point may collide with the border with the change of a sys-

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tem parameter. When that happens, there is a sudden change in the stability of the fixed point, leading to a new type of nonlinear phenomenon called *border collision bifurcation* [1,2]. It has been shown that such border collisions may lead to atypical bifurcation phenomena like transition from period-2 to period-3 or a sudden onset of chaos without undergoing the usual period doubling cascade [3].

A few different forms of such maps have been investigated to date:

(1) The map f is continuous, not differentiable at λ, and both dg/dx and dh/dx are finite. Such maps represent a class of switching circuits, and have been investigated in detail [2,4].
(2) The map f is continuous, not differentiable at λ, and there is a square root singularity (i.e., dh/dx is infinite) at one side of the border. Such maps represent the impact oscillator [1,5,6].
(3) The map f is discontinuous at λ, the derivative df/dx is also discontinuous at λ, but the value of the derivative at both sides of the border are finite. Such maps represent a class of electronic circuits [7] including the Colpitts oscillator [8], and the bifurcation theory for such maps has been developed recently [9].

In 1997 an experiment was reported, where the oscillations in the surface of a liquid held in a vertically oscillating container were observed using a laser probe, and it was found that under some conditions narrow jets are ejected from the center of the surface. It was shown that representation of this system required a map with not only slope singularity but also magnitude singularity at the border [10]. The proposed map had the form

$$x_{n+1} = \gamma x_n + \frac{\alpha x_n}{(x_n - \lambda)^2}, \quad \text{for } x_n < \lambda, \tag{2}$$

$$x_{n+1} = \beta + \frac{\rho x_n}{(x_n - \lambda)^2}, \quad \text{for } x_n > \lambda, \tag{3}$$

where α, β, ρ and λ are constants and γ is the bifurcation parameter. The graph of the map is schematically shown in Fig. 1. In this Letter we investigate the bifurcation phenomena in a map of the above form—for which no theory is currently available.

In this map, the vertical line $x = \lambda$ forms an asymptote for Eqs. (2) and (3), and a singularity occurs at this value. This asymptotic behavior occurs due to the geometric considerations in the waves—that the waveheight/wavelength cannot exceed some ratio before the wave becomes self-intersecting. While obtaining the bifurcation diagram, if x takes a value close to λ, the value of x at the next iterate is very high. The program has to account for this possibility. The bifurcation diagram obtained this way is presented in Fig. 2. As both domain and range of the piecewise smooth map are represented by the half-open set $[0, \infty)$, Fig. 2 and all subsequent bifurcation diagrams have been truncated.

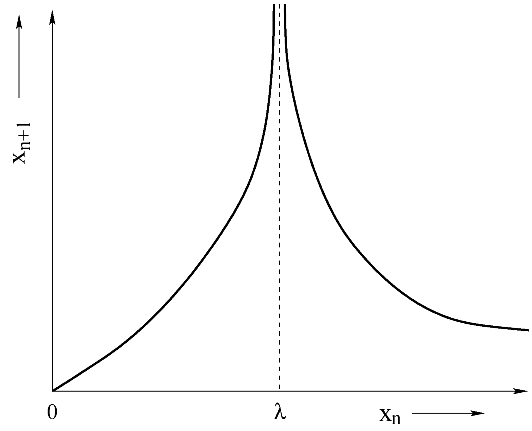

Fig. 1. The graph of the map given by (2) and (3).

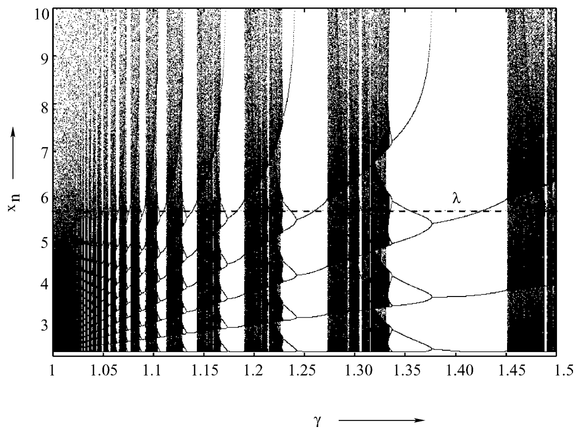

Fig. 2. Bifurcation diagram with $\alpha = 0.04$, $\beta = 2.8$, $\rho = 0.14$, $\lambda = 5.76$, and γ as the variable parameter. The diagram is truncated above $x_n = 10$.