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Near-corner waves of the Camassa–Holm equation ☆

John P. Boyd

Department of Atmospheric, Oceanic and Space Science and Program in Scientific Computing, 2455 Hayward Avenue, Ann Arbor, MI 48109, USA

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Abstract

The travelling waves of the Camassa–Holm equation exist only up to a maximum amplitude. The limiting wave has a discontinuous slope, a so-called "corner wave". Here, we investigate the transition from smooth wave to corner wave through a mixture of matched asymptotic expansions, perturbation theory and numerical computations. There are both striking similarities and equally dramatic differences from other known examples of near-corner waves.

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1. Introduction

Many species of waves exhibit the cnoidal wave/corner wave/breaking wave (CCB) scenario. That is, there are smooth spatially-periodic travelling waves ("cnoidal waves") for small amplitude, breaking waves (only) for large amplitude, and a limiting travelling wave of maximum amplitude which has a discontinuous slope—a "corner wave". A list of examples is given in [6] (and less comprehensively) in [4].

The ubiquity of the CCB scenario raises all sorts of questions. How much of the smooth-to-corner transition is generic? What qualitative details are non-generic and depend on the specific wave equation?

We have systematically studied a number of wave equations to try to answer these questions [3–6]. In this Letter, we examine the smooth-to-corner transition for the solitary waves of the Camassa–Holm equation. The pioneering article of Camassa and Holm [7] was cited more than two hundred times in the first decade after its publication. The physical and mathematical interest is explained in [1,8,10,14].

The larger goal is to understand what is generic and non-generic about the CCB bifurcation. For comparison purposes, Whitham's family of wave equations is particularly useful because these are both very simi-

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E-mail address: jpboyd@engin.umich.edu (J.P. Boyd).
URL: http://www-personal.engin.umich.edu:/~jpboyd/.

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lar and very different from the Camassa–Holm equation. We shall quote some formulas for comparison throughout the Letter, and then discuss the similarities and differences at length in Section 4.3.

2. Background on the CH equation

2.1. Implicit solution

For travelling waves, the Camassa–Holm equation is

$$cu_{XXX} + (2\kappa - c)u_X - 3uu_X - 2u_X u_{XX} - uu_{XXX} = 0, \tag{1}$$

where c is the phase speed, κ is a parameter, and the coordinate $X \equiv x - ct$. If we (i) integrate term-by-term (ii) multiply the once-integrated equation by $-2u_X$ and integrate term-by-term a second time (iii) move the non-derivative terms to the right-hand-side of the twice-integrated equation and (iv) take the square root of both sides, we obtain

$$u_X = \sqrt{\frac{(2\kappa - c)u^2 + u^3 - 2\gamma u + \mu}{(u - c)}}, \tag{2}$$

where γ and μ are constants of integration. This can be solved by the method of separation-of-variables as

$$X = \int du \sqrt{\frac{(u - c)}{(2\kappa - c)u^2 + u^3 - 2\gamma u + \mu}}. \tag{3}$$

This applies both to spatially-periodic travelling waves and solitary waves. Because cnoidal waves and solitons are very closely related [2], we shall restrict our attention to solitary waves, i.e., waves such that $|u(X)| \to 0$ as $|X| \to \infty$. For solitons, $\gamma = \mu = 0$.

For this special case, introduce the new variable [8]

$$\nu = \sqrt{\frac{c - u}{c - 2\kappa - u}} \quad \leftrightarrow \quad u = c - 2\kappa \frac{\nu^2}{\nu^2 - 1}. \tag{4}$$

Separation-of-variables then gives the solution in *implicit* form as (4) plus

$$\exp(-X) = \left\{\frac{\nu - \rho}{\nu + \rho}\right\}^{\rho} \left(\frac{\nu + 1}{\nu - 1}\right), \tag{5}$$

where

$$\rho \equiv \sqrt{\frac{c}{c - 2\kappa}} \tag{6}$$

as derived by Camassa, Holm and Hyman [8].

2.2. Theorems

The following propositions are useful in what follows:

(1) If $u(X; c', \kappa')$ is a travelling wave (soliton or non-soliton) of the CH equation, then $\sigma u(X; c, \kappa)$ is a solution with $c = \sigma c'$ and $\kappa = \sigma \kappa'$ [2].

(2) For the Camassa–Holm solitary wave (including the corner wave or "peakon"),

$$u(0) = c - 2\kappa. \tag{7}$$

(3) In the limit $\kappa/c \to 0$, the solitary wave tends to the corner wave

$$u_{\text{corner}}(X) = \exp\bigl(-|X|\bigr). \tag{8}$$

The first theorem implies that it is always sufficient to consider the case of *unit phase speed*; travelling waves of other speeds can always be obtained by applying the dilational symmetry.

The perturbation parameter that measures nearness to the corner wave is

$$\epsilon \equiv c - u(0). \tag{9}$$

Here, the parameter of the CH equation, κ, plays the same role and the second theorem shows that $\epsilon = 2\kappa$.

The third proposition furnishes an explicit, analytic form for the corner wave; this is also the "outer approximation" in the matched asymptotic approximation of the near-corner wave. It follows that the interesting and difficult challenge is to understand the "inner approximation" where the corner is rounded off when ϵ and κ are small, but not zero.

3. Matched asymptotics for near-corner waves

For Camassa–Holm solitons, surface water waves, Whitham's equation family and so on, the "outer approximation" to the near-corner wave is always the corner wave itself. This approximation fails in the inner region of width $O(\epsilon)$ where $\epsilon \equiv c - u(0)$. For all these wave equations, the inner approximation is defined by a differential equation which is most conveniently expressed in terms of the modified unknown

$$w(X) \equiv \bigl(u(X) - c\bigr)/\epsilon. \tag{10}$$