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Resonances and adiabatic invariance in classical and quantum scattering theory

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Abstract

We discover that the energy-integral of time-delay is an adiabatic invariant in quantum scattering theory and corresponds classically to an open region in phase space. The integral thus found provides a quantization condition for resonances, explaining a series of results recently found in non-relativistic and relativistic regimes. Further, a connection between statistical quantities like quantal resonance-width and classical friction has been established with a classically deterministic quantity, the stability exponent of an adiabatically perturbed periodic orbit. This relation can be employed to estimate the rate of energy dissipation in finite quantum systems.

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Wilhelm Wien [1], in one of the classic arguments, found the changes of the distribution of the energy over the spectrum and work done by a reversible adiabatic compression by employing laws of classical electrodynamics. As we know, the result was later found by purely quantum-mechanical arguments and forms a part of the Planck's law. Usage of arguments based on "response of a system to infinitesimally slow changes" was given the name, *adiabatic hypothesis* by Einstein [2]: "... If a system is exposed to adiabatic influences the 'admissible' motions are transformed into 'admissible ones'... ". There are a class of motions which become admissible by invoking adiabatic hypothesis, and in each such situation, there is a quantity that does not change before and after the adiabatic process—called an adiabatic invariant [3]. For many instructive instances from classical mechanics, the classical action turns out to be an adiabatic invariant [4]. For harmonic motions in one or more degrees of freedom, the ratio of time-averaged kinetic energy to the frequency is an adiabatic invariant [5]. In the context of statistical mechanics, entropy is an adiabatic invariant [6]. In quantum mechanics, it has been understood for long that the

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adiabatic hypothesis holds for the discrete spectrum of a Hamiltonian [7]. It was Ehrenfest [5] who, after proving that action S is an adiabatic invariant, postulated it to be quantized. Quoting Crawford [4] here would be the best: "The idea was that if a physical quantity is going to make 'all or nothing quantum jumps', it should make no jump at all if the system is perturbed gently and adiabatically, and therefore any quantized quantity should be an adiabatic invariant". In this Letter, we show the truth of this idea for the resonances of a Hamiltonian, thus in the context of quantum scattering theory. The result presented here is of fundamental importance as it not only encompasses resonances in non-relativistic and relativistic quantum mechanics but also leads us to discover relations between statistical quantities like width of a resonance, friction in a many-body system and stability exponents of the corresponding deterministic classical dynamics.

Recently, it was found that the energy-integral of time-delay, τ gives the number of resonances, n_R [8]:

$$\int_0^{E^*} \tau\, dE = n_R \hbar. \tag{1}$$

This relation has been illustrated by several examples from elementary quantum mechanics, neutron reflectometry, and high-energy physics. It has been demonstrated in a series of works starting from [9] where all the nucleon and delta resonances were reproduced by studying time-delay, ρ meson was shown in a study of time-delay for π–π scattering [10], penta-quark state has been found similarly [11].

Eq. (1) reminds one of the Bohr–Sommerfeld quantization condition for the case of bound states. For us to conclude that (1) is a quantization condition for resonances, it should be shown that the integral appearing in (1) is an adiabatic invariant.

Let us recall that the time-delay is defined by [12,13]

$$\tau = \frac{1}{N} \operatorname{Tr} Q(E), \tag{2}$$

where N is the number of channels, and $Q(E)$ is the matrix [14]

$$Q_{ij}(E) = -i\hbar \sum_{k=1}^{N} S_{ik}^* \frac{\partial S_{kj}}{\partial E}. \tag{3}$$

Here, S is the scattering matrix. Recently, an interesting expression has been found for the semiclassical time-delay in terms of open classical paths [15]. Thus, the integral in (1) is simply an integral of $Q(E)$.

For our purpose, we write $Q(E)$ in a symmetric form:

$$Q_{ij}(E) = -i\hbar \sum_k \frac{d}{d\epsilon}\left[S_{kj}\left(E + \frac{\epsilon}{2}\right) S_{ik}^*\left(E - \frac{\epsilon}{2}\right)\right]\bigg|_{\epsilon=0}. \tag{4}$$

In (1), there is only one channel, so we write

$$\begin{aligned}\int \tau\, dE = \int Q\, dE &= -i\hbar \int \frac{d}{d\epsilon}\left[S\left(E + \frac{\epsilon}{2}\right) S^*\left(E - \frac{\epsilon}{2}\right)\right]\bigg|_{\epsilon=0} dE \\ &= -i\hbar \lim_{\epsilon\to 0} \frac{d}{d\epsilon} \int S\left(E + \frac{\epsilon}{2}\right) S^*\left(E - \frac{\epsilon}{2}\right) dE. \end{aligned} \tag{5}$$

Last expression is, in fact, a derivative of two-point correlation function of S-matrix at two neighbouring energies separated by ϵ as $\epsilon \to 0$. This small parameter ϵ is in fact defining a slow, adiabatic evolution of the operator S.

We will now show that the integral in (5) is an adiabatic invariant in quantum scattering theory, with its classical analogue as an open region in phase-space bounded by a classical trajectory (see below discussion after the paragraph that follows (10)). Different from our discussion but connected with phase-space considerations is an instructive work by Narnhofer and Thirring [16].