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Theory of small aspect ratio waves in deep water

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Abstract

In the limit of small values of the *aspect ratio parameter* (or *wave steepness*) which measures the amplitude of a surface wave in units of its wave-length, a model equation is derived from the Euler system in infinite depth (deep water) without potential flow assumption. The resulting equation is shown to sustain periodic waves which on the one side tend to the proper linear limit at small amplitudes, on the other side possess a threshold amplitude where wave crest peaking is achieved. An explicit expression of the crest angle at wave breaking is found in terms of the wave velocity. By numerical simulations, stable soliton-like solutions (experiencing elastic interactions) propagate in a given velocities range on the edge of which they tend to the *peakon* solution.

Keywords: Water waves; Asymptotic methods; Nonlinear dynamics

1. Introduction

The description of the propagation of surface waves in an ideal incompressible fluid is still a classical subject of investigation in mathematical physics as no definite comprehensive answer to the problem has been given yet. In the limit of *shallow water*, surface gravity waves have been intensively studied and many model equations were introduced by various approaches, with great success. The *nonlinear deep water* case is more cumbersome and there does not exist today a simple model as universal as the shallow water equations (Korteweg-de Vries or Boussinesq) which would result from an asymptotic limit of the Euler system.

The inherent technical differences between shallow and deep water are mainly due to the fact that the two *natural* small parameters used for perturbative analysis of the Euler system in shallow water loose their sense in the deep water case (depth $h \to \infty$). Indeed, these parameter are $\epsilon_1 = a/h$, which measures the amplitude a of the perturbation scaled to the fluid depth h, and $\epsilon_2 = h^2/\lambda^2$, which measures the depth in units of wavelength λ.

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By perturbative expansion in ϵ_1, and fixed finite ϵ_2, one obtains the nonlinear shallow water equation [1]. Retaining ϵ_1 and ϵ_2 (but not their product) leads to different versions of the Boussinesq equations [2], from which the Korteweg-de Vries equation is derived by assuming a small amplitude wave moving in a given direction [3]. All these model equations govern asymptotic dynamics of long wavelength *wave profiles*.

One way to obtain a small parameter in deep water is to take into account that deep water waves typically result from a superposition of wave components close to a fundamental carrier wave. The small parameter then measures the envelope variations (scaled to the carrier wave). Such *nonlinear modulation of wave trains* is worked out perturbatively by means of *slowly varying envelope approximation* (SVEA) which usually leads, in 1+1 dimensions, to the nonlinear Schrödinger model [4,5]. For a full account on modulation of short wave trains on water of intermediate or great depth we refer to [6,7]. The procedure provides the nonlinear dynamics of surface waves as a modulation, the drawback being that the dynamics of the wave profile itself remains unknown.

Based on the theory of analytical functions and perturbation theory, a model for the profile of the free surface wave in water of finite depth, involving the Hilbert transform operator, was derived in [8]. Although this model possesses a well-defined deep water limit, the resulting equation cannot be studied by known techniques to compare it to KdV-like models. Other model equations, built to fit the properties of waves on deep water can be found in [9,10]. Their dispersion relations coincide exactly with that of the water waves on infinite depth but their nonlinear terms are chosen *ad hoc* to reproduce Stokes waves.

Our purpose here is to study the asymptotic dynamics of the very profile of a surface wave in deep water in the weak nonlinear limit by assuming a dependence on the vertical coordinate close to the linear one. The dispersion relation for a fluid of depth h

$$\omega^2 = gk\tanh(kh), \tag{1.1}$$

leads for long waves on shallow water (parameter kh small) to the nondispersive relation

$$\omega = k\sqrt{gh} \Rightarrow v_\mathrm{p} = v_\mathrm{g} = \sqrt{gh}, \tag{1.2}$$

where v_p is the phase velocity and v_g the group velocity. However, waves on deep water (parameter $kh \to \infty$) are dispersive as from (1.1)

$$\omega = \sqrt{gk} \Rightarrow v_\mathrm{p} = \sqrt{\frac{g}{k}}, \quad v_\mathrm{g} = \frac{1}{2}\sqrt{\frac{g}{k}}. \tag{1.3}$$

This *deep water dispersion relation* will be one of our main guides in the process of finding a limit model whose linear limit is constrained by (1.3).

Our approach follows the method of Green and Naghdi [11,12] for surface waves in shallow water which assumes an anzatz for the dependence of the velocity components on the vertical dimension z. This anzatz does not produce an exact solution of the full Euler system and the game consists in replacing one of the equations with its *integrated expression*. This comes actually to making an average over the depth, which can be performed with different weights. Although weight is not determinant in the shallow water case [13], we shall see that its choice is prescibed by a consistency requirement within the linear limit.

A limit model is then obtained by defining a small parameter which measures the amplitude of a surface wave in units of its wave-length, we call it the *aspect ratio parameter* (ARP), it is also referred to as the *wave steepness*. Our approach combines the asymptotic analysis *à la Whitham* with the already standard method of multiple scales [14–18], it will be shown to lead to the following model

$$\eta_t - \eta_{xxt} - \frac{1}{2}\eta_{xxx} + \frac{3}{2}\eta_x + \eta\eta_x = \frac{5}{3}\eta_x\eta_{xx} + \frac{1}{3}\eta\eta_{xxx}, \tag{1.4}$$

for the dimensionless deformation $\eta(x, t)$ of the deep water free surface.

The paper is organized as follows. In Section 2, we introduce the Euler equations, their nondimensional version and the anzatz which, together with a convenient average, enables to reduce the initial three-dimensional problem