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Rotation number of the overdamped Frenkel–Kontorova model with ac-driving

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Abstract

The dynamical behavior of the overdamped and overdamped limit cases of the Frenkel–Kontorova model with ac-driving is studied. The relationship is established between the average velocity of the particles and the rotation number of the orientation preserving homeomorphism on the circle induced from the Poincaré map. The dynamical behavior for both cases is determined completely by the rotation number. Numerical results are also provided for the second order system to support the theoretical conclusions.

Keywords: Overdamped Frenkel–Kontorova model; Average velocity; Orientation preserving homeomorphism; Rotation number; Mode-locking

1. Introduction

The standard Frenkel–Kontorova model describes the dynamics of a chain of particles interacting with the nearest neighbors in the presence of an external periodic potential. In fact, many physical systems can be analyzed effectively with the help of the Frenkel–Kontorova model and its generalizations, see chapter 2 of [1]. For example, charge-density waves and Josephson junction arrays can be described by the overdamped Frenkel–Kontorova model [4]. In this paper, we consider the overdamped limit and overdamped cases of the standard Frenkel–Kontorova model

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with ac-driving:

$$\dot{u}_j = K(u_{j+1} + u_{j-1} - 2u_j) - \frac{1}{2\pi}\sin(2\pi u_j) + F(t), \tag{1.1}$$

and

$$\varepsilon\ddot{u}_j + \dot{u}_j = K(u_{j+1} + u_{j-1} - 2u_j) - \frac{1}{2\pi}\sin(2\pi u_j) + F(t), \tag{1.2}$$

where $\varepsilon > 0$, $K > 0$ and $\varepsilon < 1/4(2K+1)$, and $F(t) = \overline{F} + F_{\mathrm{ac}}\cos 2\pi\nu_0 t$ with period $T = 1/\nu_0$.

The most important general result for the overdamped limit Eq. (1.1) is the "no passing" rule, or "strong monotonicity" by mathematician, from which many consequences have been derived.

For system (1.1), when the driving force is constant, Baesens and MacKay [2] gave complete proofs for some claims of Middleton [7] and of Floría and Mazo [4] by making use of the "strong monotonicity" of the solutions to (1.1). They then extended their method to system (1.2) in [3] provided the driving force $F(t)$ is constant and the overdamped condition is satisfied, i.e., $\varepsilon < 1/4(2K+1)$. The main idea of [3] is to construct a partial order which is an analogue of the monotonicity for the first order system and is preserved by system (1.2) under overdamped condition. In fact, the partial order preserved by the overdamped second order system can be traced to 1988, in which Qian et al. in [8] and Levi in [6] found simultaneously a partial order which is preserved by an overdamped single oscillator, i.e., by (1.2) with $K = 0$ and $\varepsilon < 1/4$. A monotonicity for two coupled oscillators was discussed in [9].

In the present paper, we are concerned with the dynamical behavior of the overdamped Frenkel–Kontorova model with the average spacing being rational and the driving force $F(t)$ being time dependent. For the first order system (1.1) with a sinusoidal force, many interesting phenomena were observed in numerical simulations for commensurate structures, such as mode-locking behavior and unlocking transition. In Section 2 we prove the following conclusions for (1.1) with periodic boundary condition: the existence and uniqueness of the average velocity $\overline{v}$, and the continuity and non-decreasing property of $\overline{v}$ with respect to the average of the external driving force. With these theoretical results, we explain and discuss the mode-locking phenomenon reported in [4]. In Section 3 we prove that the conclusions for the first order system hold true for the overdamped second order system with periodic boundary condition. We also provide numerical results which coincide with the theoretical conclusions. Our main idea is as follows. Rather than directly making use of the strong monotonicity mentioned in [2] and found in [3], we make a change of variables and specify monotonicity (strong monotonicity) in new coordinate system, which, of course, is equivalent to that in [2,3]. Then we define a concept of "horizontal curve". The monotonicity implies that the image under the Poincaré map P^{T} of a horizontal curve is still a horizontal curve. The necessity of making variables transformation will be explained in Section 4. By virtue of periodicity of the vector field and Arzela–Ascoli theorem we construct a Banach space such that the set of the bounded horizontal curves corresponds to a closed convex relatively compact subset Ω of the Banach space and $\hat{P}^{\mathrm{T}}(\Omega) \subset \Omega$, in which $\hat{P}^{\mathrm{T}}$ is an induced map from the Poincaré map. From Schauder fixed point theorem it follows that P^{T} has an invariant horizontal curve ℓ and the action of P^{T} on ℓ is exactly an orientation preserving homeomorphism on the unit circle. Then we discuss the dynamical behavior of systems (1.1) and (1.2) with periodic boundary conditions and establish the relationship between the average velocity of the particles and the rotation number of the induced orientation preserving circle map. We should mention here that the approach developed in [2,3] cannot be extended directly to the cases with ac-driving, which is one of the open questions listed in [2]. However, the approach proposed in this paper works naturally for the case with constant driving force. Indeed, one can easily deduce that the invariant curve obtained for finite chains is exactly a periodic sliding solution if no equilibrium point exists.