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A variational approach to singular perturbation problems in reaction–diffusion systems

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Abstract

In this paper singular perturbation problems in reaction–diffusion systems are studied from a viewpoint of variational principle. The goal of the study is to provide an unified and transparent framework to understand existence, stability and dynamics of solutions with transition layers in contrast to previous works in many literatures on singular perturbation theory.

Keywords: Singular perturbation; Transition layer; Reduced energy

1. Introduction

In various physical phenomena there are many interesting patterns that consists of two distinct phases such as solid phase and liquid phase in crystal growth. In order to study such patterns, we focus on a sharp transition layer between two different phases, which is called an interface. The phenomena that exhibit patterns with transition layers can be often described by differential equations with small parameters which characterize the width of a narrow region of the transitions. The singular perturbation problem arising in the study of properties for solutions to such differential equations has interested many researchers, and there are huge number of literatures concerning this topic. For example, the boundary layer problem raised in fluid mechanics, and internal transition layers in pattern formation in dissipative systems, and so on. We refer to standard texts [21,22,35] and the references therein.

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In singular perturbation problems we often encounter rather lengthy and involved calculations based on ad hoc ideas. It is usually a hard task to follow such calculations except specialists, as a consequence, difficult to understand a mathematical meaning behind technical calculations. The purpose of this paper is to give an unified and transparent approach to singular perturbation problems arising in reaction–diffusion systems from a viewpoint of variational principle in contrast to previous works in literatures. In order to explain our motivation and idea as clear as possible, we consider a simple one-dimensional reaction–diffusion equation

$$u_t = \varepsilon^2 u_{xx} + f(u, x), \quad 0 < x < 1, \tag{1.1}$$

under the Neumann boundary condition $u_x = 0$. A typical example of nonlinear term f is

$$f(u, x) = \frac{1}{2}(1 - u^2)(u - a(x)), \tag{1.2}$$

where $a(x)$ is a smooth function satisfying $-1 < a(x) < 1$. For the present, we suppose that f is given by (1.2) though the theory presented in this paper is valid for more general nonlinear terms. (1.1) admits a gradient (variational) structure

$$u_t = -\frac{\delta \mathcal{E}[u]}{\delta u}, \tag{1.3}$$

where $\mathcal{E}[u]$ is an energy functional defined by

$$\mathcal{E}[u] = \int_0^1 \left\{ \frac{\varepsilon^2}{2} u_x^2 - F(u, x) \right\} \mathrm{d}x, \tag{1.4}$$

and $F_u(u, x) = f(u, x)$. We easily see that an energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}[u(x, t)] = -\int_0^1 u_t^2(x, t)\, \mathrm{d}x \tag{1.5}$$

holds. Moreover, it is well known [2,4,13,18,30] that (1.1) possesses solutions with sharp transition layers for sufficiently small ε. Here, for simplicity, we consider solutions with a single transition layer as seen in Fig. 1.

First, we are concerned with stationary solutions of (1.1) as in Fig. 1, which are defined by

$$0 = \varepsilon^2 u_{xx} + f(u, x), \quad 0 < x < 1. \tag{1.6}$$

Although we can construct such solutions by using the traditional method known as matched asymptotic expansion technique, we adopt a different appoach based on the variational structure (1.3). We divide the whole interval $(0, 1)$ into two subintervals $(0, \ell)$ and $(\ell, 1)$, and consider solutions of (1.6) on each subinterval as follows:

$$\begin{cases} \varepsilon^2 \phi_{1xx} + f(\phi_1, x) = 0, \quad 0 < x < \ell, \\ \phi_{1x}(0) = 0, \quad \phi_1(\ell) = 0, \end{cases}$$

and

$$\begin{cases} \varepsilon^2 \phi_{2xx} + f(\phi_2, x) = 0, \quad \ell < x < 1, \\ \phi_{2x}(1) = 0, \quad \phi_2(\ell) = 0. \end{cases}$$