Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physica D 207 (2005) 52–63
PHYSICA D
www.elsevier.com/locate/physd

On the creation of Wada basins in interval maps through fixed point tangent bifurcation

Romulus Breban[a,*], Helena E. Nusse[b,1]

[a] *Department of Physics, Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, MD 20742, USA*
[b] *Institute for Physical Sciences and Technology, University of Maryland, College Park, MD 20742, USA*

Received 5 April 2004; received in revised form 10 May 2005; accepted 17 May 2005
Available online 13 June 2005
Communicated by C.K.R.T. Jones

Abstract

Basin boundaries play an important role in the study of dynamics of nonlinear models in a variety of disciplines such as biology, chemistry, economics, engineering, and physics. One of the goals of nonlinear dynamics is to determine the global structure of the system such as boundaries of basins. A basin having the strange property that every point which is on the boundary of that basin is on the boundary of at least three different basins, is called a *Wada basin*, and its boundary is called a *Wada basin boundary*. Here we consider maps on the interval. We present a sufficient and necessary condition guaranteeing that three Wada basins are emerging from a tangent bifurcation for certain one-dimensional maps having negative Schwarzian derivative, two fixed point attractors on one side of the tangent bifurcation, and three fixed point attractors on the other side of the tangent bifurcation. All the conditions involved are numerically verifiable.

PACS: 05.45.-a; 05.45.Df; 02.30.Oz

MSC: 37E05

Keywords: Wada basin boundary; Saddle-node bifurcation; Interval map

1. Introduction

In nonlinear dynamics, there are often two or more attractors. A *basin* of attraction is the collection of points whose trajectories approach a specified compact invariant set such as an attractor. A point x is a *boundary point* of a basin B if every open neighborhood of x has a nonempty intersection with basin B and at least

* Corresponding author at: Department of Biomathematics at David Geffen School of Medicine, University of California at Los Angeles, Penthouse 2, 1100 Glendon Avenue, Los Angeles, CA 90024, USA. Tel.: +1 310 794 8650.
E-mail address: breban@ucla.edu (R. Breban).
[1] Present address: Department of Econometrics, University of Groningen, P.O. Box 800, NL-9700 AV, Groningen, The Netherlands.

0167-2789/$ – see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2005.05.012

one other basin. The *boundary* of a basin is the set of all boundary points of that basin. For a basin B, we write $\partial\overline{B}$ for the boundary of B. (Note that this definition is slightly different from the topological definition of boundary ∂B.) One of the goals of nonlinear dynamics is to determine the global structure of a system such as boundaries of basins.

One can imagine situations for which a boundary point of a basin is on the boundary of at least two other basins. Examples may suggest that there are only finitely many such points. Imagine trying to create a picture of three nonoverlapping regions in the plane, each connected, which have the property that every boundary point of any region is a boundary point of all three regions. Such regions can be created and a construction was given early on by the Dutch mathematician Brouwer [1]. Independently, Yoneyama gave in 1917 an example comparable to Brouwer's example leading to the result that every boundary point is a boundary point of three open sets. Yoneyama attributed the example to "Mr. Wada". In the book by Hocking and Young [2], this example is entitled "Lakes of Wada". As originally presented, these examples are not related to dynamical systems. It is hard to imagine that such a configuration of three basins could exist for simple dynamical processes. Kennedy and Yorke [3] discovered that such "Wada basins" might occur in some simple processes.

Indeed, a boundary point of a basin B may be a boundary point of at least two other basins. Following [4,5] we say that a point x is a *Wada point* if every open neighborhood of x has a nonempty intersection with at least three basins. A basin B is called a *Wada basin* if every boundary point of B is a Wada point; the boundary of a Wada basin is called a *Wada basin boundary*. In other words, if you zoom in on a boundary point, no matter how close, all three basins would be in the detailed picture. A system has the *Wada property* if the map has at least three basins of attraction and if all basin boundaries coincide. This definition was given by Kennedy and Yorke [3]. They were unable to prove that the Wada property occurs for basins except in rather special circumstances. However, based on pictures of basins, Kennedy and Yorke argued that the Wada property appears to exist even in the forced damped pendulum. General criteria guaranteeing the existence of Wada basins for two-dimensional systems are presented in [6]. Wada basins occur in a variety of systems and recent papers discussing Wada basins are [7–11].

In refs. [12,10] it is argued why Wada basins emerge when a saddle-node bifurcation occurs on a fractal basin boundary in a variety of two-dimensional systems like the forced damped pendulum or the forced Duffing oscillator. In these references, it is assumed that the saddle-node bifurcation occurs on a fractal boundary. In this paper, we investigate whether Wada basins can emerge when a tangent bifurcation occurs for maps on a compact interval. We say basin B has a *fractal basin boundary* if its boundary is a Cantor set. Examples with fractal basin boundaries are common and have been studied extensively, see e.g. [13,14] and references therein. Note that a Wada basin boundary is a fractal basin boundary but a fractal basin boundary need not be a Wada basin boundary. Furthermore, we point out that if the map has the Wada property then every basin is a Wada basin, but the fact that every basin is a Wada basin does not imply that the map has the Wada property.

In the bifurcation theory literature, local bifurcations such as tangent bifurcations for one-dimensional maps are usually studied from a local point of view; see e.g. [15–19]. For example, in the tangent bifurcation theory for maps, attention has focused on the local stability properties of the two fixed points (or periodic points) of differentiable maps when a tangent bifurcation occurs. We say that a one parameter family of maps f_μ has at the critical parameter value μ_0 a *fixed point (periodic point) creating tangent bifurcation* at the location x_0, if the map f_μ has no fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 - \epsilon < \mu < \mu_0$, and f_μ has two fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 < \mu < \mu_0 + \epsilon$, where $0 < \epsilon \ll 1$; see, for example, Fig. 3 below. Similarly, we say that a one parameter family of maps f_μ has at the critical parameter value μ_0 a *fixed point (periodic point) destroying tangent bifurcation* at the location x_0 if the map f_μ has two fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 - \epsilon < \mu < \mu_0$, and f_μ has no fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 < \mu < \mu_0 + \epsilon$, where $0 < \epsilon \ll 1$. We say that a one parameter family of maps f_μ has at the critical parameter value μ_0 a *tangent bifurcation* at the location x, if it is either a fixed point (periodic point) creating tangent bifurcation