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A variational approach to the stability of an embedded NLS soliton at the edge of the continuum

A.A. Minzoni[a], Noel F. Smyth[b,*], Annette L. Worthy[c]

[a] *FENOMEC, Department of Mathematics and Mechanics, I.I.M.A.S., Universidad Nacional Autónoma de México, Apdo. 20–726, 01000 D.F., México*
[b] *School of Mathematics, The King's Buildings, University of Edinburgh, Edinburgh, Scotland, EH9 3JZ, UK*
[c] *School of Mathematics and Applied Statistics, University of Wollongong, Northfields Avenue, Wollongong, NSW 2522, Australia*

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Abstract

A Lagrangian-based method is used to construct embedded soliton solutions of a nonlinear Schrödinger equation with higher order dispersive terms. It is shown that the embedded soliton consists of two widely separated peaks with linear, dispersive radiation of exponentially small amplitude between the peaks, this radiation acting to hold the pulses together. The Lagrangian-based technique determines the details of this embedded soliton, including the amplitude of the pulses, the separation of the pulses and the amplitude of the radiation. In addition, it is found that the embedded soliton exists for only discrete eigenvalues, related to the separation of the pulses. Moreover, since the embedded soliton solutions are steady, they are at the lower edge of the continuous spectrum. Finally, by allowing the parameters of the embedded soliton to be time dependent, it is found that the embedded soliton develops an oscillatory type of one-sided stability. This one-sided stability is different from the usual monotone one-sided stability found for solitons embedded inside the continuous spectrum.

Keywords: Soliton; Embedded soliton; NLS equation; Variational; Modulation theory

1. Introduction

An interesting and unusual aspect of the major role played by radiation in the evolution of solitary waves is the resonant interaction between a moving solitary wave and radiation or the resonant interaction between a standing, periodic solitary wave and radiation. In the case of a moving solitary wave, this resonant interaction arises from

* Corresponding author. Tel.: +44 131 650 5080; fax: +44 131 650 6553.
E-mail addresses: tim@mym.iimas.unam.mx (A.A. Minzoni); N.Smyth@ed.ac.uk (N.F. Smyth); Annette_Worthy@uow.edu.au (A.L. Worthy).

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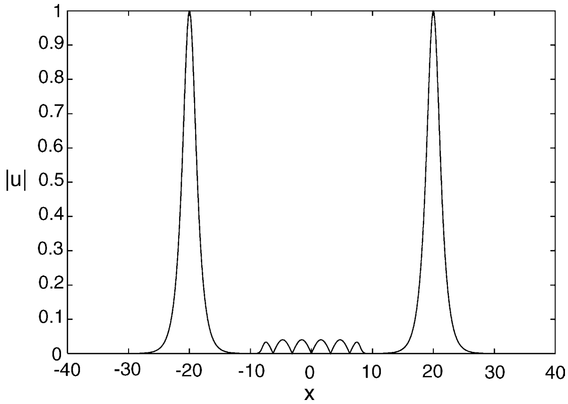

Fig. 1. Two-pulse, embedded soliton.

the embedding of the solitary wave velocity in the continuous (radiation) spectrum [1]. This resonant interaction vanishes at a specific value of the velocity, resulting in a so-called embedded soliton [1]. Modulated solutions around an embedded soliton were studied in [2]. It was shown that a momentum conservation argument, coupled with the fact that the radiated momentum is mostly confined in a region expanding with the group velocity for the resonant wavenumber, gives a quadratic modulation equation for the velocity. This quadratic modulation equation in turn implies a one-sided stability for the embedded soliton, that is if the initial amplitude is below the steady value, the embedded soliton is unstable, while if it is above the steady value, the embedded soliton is stable. For the case of a standing solitary wave with a frequency embedded in the continuous spectrum, a similar analysis [1] again gives a one-sided stability for the embedded soliton. In addition the case of a family of stable embedded solitons has been studied by this method and the resulting modulation equations show stability [3].

Multi-pulse embedded solitons can also be determined for a large class of equations [4]. These multi-pulse embedded solitons are held together by exponentially small wavetrains. Such a multi-pulse embedded soliton is given schematically in Fig. 1. Moreover the analysis of [1] could again be used to determine the one-sided stability of these multi-pulse solutions, since the frequency of oscillation of the solitary wave is embedded within the continuous spectrum. However, to the best knowledge of the authors, this analysis has not been carried out. An asymptotic analysis of the stability of a steady embedded soliton has not as yet been carried out.

The present work considers the nonlinear Schrödinger (NLS) equation in the limit of zero dispersion, so that higher order dispersive terms become important, as in [5]. This work can be regarded as complementary to that of [1,2,4,6,7] as the question of the stability of a two-humped, steady embedded soliton is addressed and resolved using a modulation theory. For this perturbed NLS equation, since the component solitary waves have zero frequency, the potentially excited linear, dispersive waves have zero wavenumber and zero group velocity. A degenerate case is therefore considered for which the solitary waves are embedded at the edge of the continuous spectrum. Because of this potentially resonant zero wavenumber wave, we represent the resonance, as in previous work [8–10], by modifying the trial function of [5] to include a shelf (of radiation). This representation of the radiation, which is needed to handle the resonance, must be coupled to a modulated double-humped solution. This situation then leads to a variational formulation to determine the steady embedded soliton. This variational formulation treats simultaneously the double humps and the radiation which links them, unlike [4,6], in which the humps and linking radiation are determined in two sequential steps. The present variational formulation is in terms of a free boundary which couples the radiation to the humps. This free boundary is determined in the extremisation.

It is shown that the variational procedure leads to modulation equations whose dynamics are controlled by the evolution of the slow phases of the double humps (solitary waves). It is further shown that the steady state is a degenerate centre with zero frequency, since the standing solution, unlike the cases considered in [1,4], has zero