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Regular dynamics in a delayed network of two neurons with all-or-none activation functions

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Abstract

We consider a delayed network of two neurons with both self-feedback and interaction described by an all-or-none threshold function. The model describes a combination of analog and digital signal processing in the network and takes the form of a system of delay differential equations with discontinuous nonlinearity. We show that the dynamics of the network can be understood in terms of the iteration of a one-dimensional map, and we obtain simple criteria for the convergence of solutions, the existence, multiplicity and attractivity of periodic solutions.

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1. Introduction

We consider the following model for an artificial network of two neurons

$$\begin{cases} \dot{x} = -\mu x + a_{11} f(x(t-\tau)) + a_{12} f(y(t-\tau)), \\ \dot{y} = -\mu y + a_{21} f(x(t-\tau)) + a_{22} f(y(t-\tau)), \end{cases} \tag{1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $x(t)$ and $y(t)$ denote the state variables associated with the neurons, $\mu > 0$ is the interact decay rate, $\tau > 0$ is the synaptic transmission delay, a_{11}, a_{12}, a_{21} and a_{22} are the synaptic weights, and $f : \mathbb{R} \to \mathbb{R}$ is the

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activation function. Such a model describes the evolution of the so-called Hopfield net [6,7,15] where each neuron is represented by a linear circuit consisting of a resistor and a capacitor, and where each neuron is connected to another via the nonlinear activation function f multiplied by the synaptic weights $a_{ij}(i \neq j)$. We also assume that each neuron has self-feedback and signal transmission is delayed due to the finite switching speed of neurons.

We focus here on the computational performance described by the asymptotical behaviors of model (1), where the signal transmission is of digital nature: a neuron is either fully active or completely inactive. Namely, the signal transmission is of McCulloch–Pitts type [8–10,16,17,21] and we have

$$f(\xi) = \begin{cases} -\delta, & \text{if } \xi > 0, \\ \delta, & \text{if } \xi \leq 0, \end{cases} \tag{2}$$

where $\delta > 0$ is a given constant. Therefore, the model describes a combination of analog and digital signal processing. Differential equations of this type usually occur in control systems, e.g., in heating systems and the pupil light reflex, if the controlling function is determined by a constant delay $\tau > 0$ and the switch recognizes only the positions "*on*" $[f(\xi) = \delta]$ and "*off*" $[f(\xi) = -\delta]$. Because each variable changes continuously but depends on the signs of other variables, such a system retains a continuous-time framework and can be proposed as a useful simplification to gain analytical insight (see, for example [4]). In addition, a rather confusing variety of names have been applied to this system, such as "Glass networks"(see, for example [4,5]), "piecewise-linear equations", "switching networks", "nonlinear chemical reaction networks", "gene networks", "Boolean kinetic equations" and variants of these. Here we avoid this confusion by calling it "McCulloch–Pitts networks". By the discontinuous nonlinearity, the differential equation allows detailed analysis. It turns out that there is a rich solution structure. To simplify our presentation, we first rescale the variables by

$$t^* = \mu t, \quad \tau^* = \mu\tau, \quad x^*(t^*) = \frac{\mu}{\delta}x(t), \quad y^*(t^*) = \frac{\mu}{\delta}y(t), \quad f^*(\xi) = \frac{1}{\delta}f\left(\frac{\delta}{\mu}\xi\right),$$

and then drop the $*$ to get

$$\begin{cases} \dot{x} = -x + a_{11}f(x(t-\tau)) + a_{12}f(y(t-\tau)), \\ \dot{y} = -y + a_{21}f(x(t-\tau)) + a_{22}f(y(t-\tau)) \end{cases} \tag{3}$$

with

$$f(\xi) = \begin{cases} -1, & \text{if } \xi > 0, \\ 1, & \text{if } \xi \leq 0. \end{cases} \tag{4}$$

It is natural to have the phase space $X = C([-\tau, 0]; \mathbb{R}^2)$ as the Banach space of continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^2$ equipped with the sup-norm, see [13]. Note that for each given initial value $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X$, one can solve system (3) on intervals $[0, \tau]$, $[\tau, 2\tau]$, ... successively to obtain a unique mapping $(x^\Phi, y^\Phi)^{\mathrm{T}}$: $[-\tau, \infty) \to \mathbb{R}^2$ such that $x^\Phi|_{[-\tau,0]} = \varphi$, $y^\Phi|_{[-\tau,0]} = \psi$, $(x^\Phi, y^\Phi)^{\mathrm{T}}$ is continuous for all $t \geq 0$, piecewise differentiable and satisfies (3) for $t > 0$. This gives a unique solution of (3) defined for all $t \geq -\tau$. In applications, a network usually starts from a constant (or nearly constant) state. Therefore, we shall concentrate on the case where each component of Φ has no sign change and has at most finitely many zeros on $[-\tau, 0]$. More precisely, we consider $\Phi \in X^{+,+} \bigcup X^{+,-} \bigcup X^{-,+} \bigcup X^{-,-} = X_0$, where

$$C^{\pm} = \{\pm\varphi; \varphi : [-\tau, 0] \to [0, \infty) \text{ is continuous and has only finitely many zeros on } [-\tau, 0]\}$$

and

$$X^{\pm,\pm} = \{\Phi \in X; \quad \Phi = (\varphi, \psi)^{\mathrm{T}}, \varphi \in C^{\pm} \text{ and } \psi \in C^{\pm}\}.$$

Clearly, all constant initial values (except for 0) are contained in X_0.