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Some properties of the magnetic fields generated by symmetric configurations of wires

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Abstract

In this paper we study some properties of the magnetic field lines and their effect on the particle motions. For certain configurations of wires we prove the existence of first integrals and ergodic (quasi-periodic) orbits. When the magnetic fields possess a Euclidean symmetry we prove that the equations of motion (in the relativistic and non-relativistic cases) inherit a first integral different from the kinetic energy. As a consequence of this property we show that in some physical examples the wires creating the magnetic field are unreachable for electric charges and there exist confinement spatial regions. Part of these mathematical results are of interest to electrical engineers, helping to keep the power lines electrically neutral.

Keywords: Magnetic field; Newton equation; First integral; Symmetry

1. Introduction

Consider a (regular) curve L parametrized by the map $\tau : R \to R^3$, that is $L = \{\mathbf{x} = \tau(s)\}$. This curve can be bounded or unbounded. If we associate a constant current J to this curve we have an electric wire (L, J). In what follows we will consider that J is positive when the current flows across the wire in the direction of the tangent vector $\mathrm{d}\tau/\mathrm{d}s$. The magnetic field $\mathbf{B}$ created by L is defined by the Biot-Savart law [1] (in cartesian coordinates $\mathbf{x} = (x, y, z)$)

$$\mathbf{B} = \int_{-\infty}^{+\infty} J \frac{\dot{\tau}(s) \wedge (\mathbf{x} - \tau(s))}{\|\mathbf{x} - \tau(s)\|^3}\,\mathrm{d}s \tag{1}$$

where the dot over τ stands for the derivative with respect to s and $\wedge$ and $||\ ||$ stand for the standard vector product and Euclidean norm in R^3. It is well known that this vector field is divergence-free ($\mathrm{div}(\mathbf{B}) = 0$) [1].

It is immediate to obtain the magnetic field $\mathbf{B}$ created by a (finite) configuration $C = \bigcup_i L_i$ of different wires via linear superposition, that is $\mathbf{B} = \sum_i \mathbf{B}_i$. Accordingly $\mathbf{B}$ is a vector field defined in the whole R^3 except in the singular set C.

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In this paper we are interested in the geometrical properties of **B** induced by the configuration C of wires. Specifically we study the symmetry vector fields **S** ($[\mathbf{B}, \mathbf{S}] = 0$, [] standing for the Lie-Jacobi bracket) and the first integrals I ($\mathbf{B}(I) = 0$) of magnetic fields created by different configurations. Note that ergodicity and chaos are compatible with the restriction $\mathrm{div}(\mathbf{B}) = 0$ [2–4] and therefore the orbit structure of **B** can be quite tangled.

Symmetries and first integrals are very important tools in order to avoid chaotic behaviour [5] and to study the existence of closed and ergodic (quasi-periodic) orbits of **B**. Recall that in the Physics literature the level sets of the first integral I of **B** are called *magnetic surfaces*. Note that the orbits of the magnetic field are tangent to the magnetic surfaces because the level sets I = constant are invariant under the flow of **B** (i.e. $\mathbf{B}(I) = 0$).

The orbit structure of the magnetic fields created by configurations of wires is poorly studied in the literature. Apart from very simple configurations (the straight and circular wires) [1], [6–8] there is a lack of rigorous results concerning this topic. Note that in reference [9] the study of first integrals of **B** is suggested (it is raised a conjecture concerning the algebraicity of I), although interesting or general results on this topic are not obtained. In this work we fill this gap by studying the phase portrait of the magnetic fields generated by different symmetric configurations.

On the other hand the knowledge of the qualitative properties of the orbits of **B** is important to understand the motion of charged particles subjected to this field. For a unit mass, unit charge particle the Newton equations read

$$\ddot{\mathbf{x}} = \dot{\mathbf{x}} \wedge \mathbf{B}(\mathbf{x}) \tag{2}$$

where in this case the dot represents the derivative with respect to the time variable t.

Eq. (2) always possesses a first integral, the kinetic energy,

$$T = \tfrac{1}{2}\dot{\mathbf{x}}^2. \tag{3}$$

Note that when Eq. (2) is substituted by the corresponding relativistic equation then the kinetic energy takes the form

$$T = \gamma - 1 \equiv (1 - \dot{\mathbf{x}}^2)^{-1/2} - 1. \tag{4}$$

It is an interesting problem to study how the geometrical structure of the orbits of **B** affects the particles motion. For example, if the magnetic field possesses a first integral, whose level sets are magnetic surfaces, do new conservation laws appear in the charges motion? Do confinement of the particles exist? What qualitative properties of the motion can be deduced from the phase portrait of **B**?

Understanding the influence of the magnetic surfaces on the motion of charge particles is of great importance in Physics, for instance in the study of trapping and evolution of plasmas [10].

To the best of our knowledge most of the approaches to this problem are perturbative, for example the drift or guiding-centre approximation [11–13] (see also [1]). In reference [14] a force-free magnetic field, for which this approximation fails, is studied. The authors show that the motion of charged particles subjected to this field is integrable and, as consequence of this fact, some qualitative properties of their orbits are obtained. Apart from this result, which is very specific and not related to any physical system, we have not been able to trace exact or global results back in the literature. In this paper we prove that when **B** possesses certain symmetries the particles motion inherits additional first integrals and is confined to certain spatial regions. These results fill an existing gap in the literature of this topic and could be useful in the Physics of plasmas, where confinement of particles is an important issue.

Summarizing, the organization of this paper is as follows:

- In Section 2 we study magnetic fields possessing a *Euclidean symmetry*. We prove that the equations of motion inherit a conservation law. These results are applied to different realistic configurations and we show that the wires creating a magnetic field **B** are unreachable for electric charges which move under the action of **B**. The interest of this fact for physical applications is notorious: it gives a formal justification to various phenomena that appear in electrical engineering, such as the maintenance of neutrality in power lines.
- In Section 3 we consider systems with a *radial symmetry*. We prove that near the wires, the field lines are small meridional circles. The confinement of particles subjected to this field is also discussed.