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Robust heteroclinic cycles in the one-dimensional complex Ginzburg–Landau equation

D.J.B. Lloyd*, A.R. Champneys, R.E. Wilson

Bristol Centre for Applied Nonlinear Mathematics and Bristol Laboratory for Advanced Dynamics Engineering, Department of Engineering Mathematics, University of Bristol, Queen's Building, University Walk, Bristol BS8 1TR, UK

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Abstract

Numerical evidence is presented for the existence of stable heteroclinic cycles in large parameter regions of the one-dimensional complex Ginzburg–Landau equation (CGL) on the unit, spatially periodic domain. These cycles connect different spatially and temporally inhomogeneous time-periodic solutions as $t \to \pm\infty$. A careful analysis of the connections is made using a projection onto five complex Fourier modes. It is shown first that the time-periodic solutions can be treated as (relative) equilibria after consideration of the symmetries of the CGL. Second, the cycles are shown to be robust since the individual heteroclinic connections exist in invariant subspaces. Thirdly, after constructing appropriate Poincaré maps around the cycle, a criteria for temporal stability is established, which is shown numerically to hold in specific parameter regions where the cycles are found to be of Shil'nikov type. This criterion is also applied to a much higher-mode Fourier truncation where similar results are found. In regions where instability of the cycles occurs, either Shil'nikov–Hopf or blow-out bifurcations are observed, with numerical evidence of competing attractors. Implications for observed spatio-temporal intermittency in situations modelled by the CGL are discussed.

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1. Introduction

Heteroclinic cycles have been observed and analysed in a variety of PDEs including the Kuramoto–Sivashinsky equation and Navier–Stokes equations, see [17]. Such cycles are characterised by metastable, recurrent behaviour,

* Corresponding author.
E-mail addresses: david.lloyd@bris.ac.uk (D.J.B. Lloyd), a.r.champneys@bris.ac.uk (A.R. Champneys), re.wilson@bris.ac.uk (R.E. Wilson).

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made up of long periods of quasi-static regimes with sudden bursts of aperiodic, spatio-temporal evolution, 'relaxing' after a while to new quasi-static states. Systems possessing symmetries are often found to admit *robust* heteroclinic cycles that persist under perturbations that respect the symmetry of the system. For example, codimension-two mode interactions in systems with O(2) symmetry, are known to provide a rich variety of robust heteroclinic cycles between equilibria and/or periodic solutions, see [1,28,7,23].

The one-dimensional complex Ginzburg–Landau (CGL) equation on a periodic domain is given by

$$u_t = (1 + \mathrm{i}\nu)u_{xx} + Ru - (1 + \mathrm{i}\mu)|u|^2u, \tag{1}$$

where $u \in \mathbb{C},\ \nu, \mu, R \in \mathbb{R},\ x \in [0, 1]$ periodic. It can be shown to be the generic amplitude equation on long space and time scales close to the critical Reynolds number for spatio-temporal pattern formation in fluid dynamics, see Newell et al. [31]. More generally, the CGL can be thought of as a normal form for a Hopf bifurcation in a variety of spatially extended systems. The CGL has been used to study many practical problems such as chemical turbulence, Poiseuille flow, Taylor–Couette flow, and Rayleigh–Bénard convection; see Mielke [25] for a review.

Numerous analytical and numerical investigations of the CGL with periodic boundary conditions have been carried out. Analytical results have concentrated on bifurcations from the trivial solution where new solutions can be found from reductions of the CGL to an ODE, see [12]. A closed form solution to the CGL for arbitrary initial data is not known and so numerical investigations provide the only way to fully explore its dynamics away from analytically known special solutions. There have been a few bifurcation sequences mapped out for $\nu = -\mu$, $R = 0, \ldots, 100$ ([11,16,26,21]). However, this paper is concerned with exploration away from the line $\nu = -\mu$, where we shall find wide parameter regions where robust heteroclinic cycles occur.

Rodriguez and Schell [33], analysed a two-mode Fourier truncation in an invariant subspace of the CGL but only found structurally unstable heteroclinic cycles. The heteroclinic cycles that we observe in the full PDE are structurally stable (i.e., they persist under perturbations that respect the symmetry of the CGL) and are not described by the truncation of [33]. We find that the minimal truncation necessary to observe these cycles is five complex Fourier modes, in which setting we carry out a Shil'nikov-type analysis which shows why the cycles should be robust.

In this paper we will describe and explain the existence and stability of robust heteroclinic cycles in the one-dimensional, complex Ginzburg–Landau equation posed on the spatially periodic domain. Our approach to the problem is similar to that of Rucklidge and Matthews [34] who analysed two-dimensional PDEs governing magnetoconvection. We tackle these heteroclinic cycles in an intuitive manner based on observations made in the full PDE rather than looking a priori at heavy restrictions/truncations and seeing where they apply to the CGL. A variety of numerical and analytical techniques are used to gain insight into the dynamics associated with these heteroclinic cycles. We start with a minimal Fourier truncation which possesses the same symmetries as the heteroclinic cycles. Then we use Poincaré return map analysis within this truncated system to find an analytical criterion for asymptotic stability of the heteroclinic cycles and predict analytically what happens when the heteroclinic cycle loses stability. Numerical continuation is then used to explore the heteroclinic cycles observed in the Fourier truncation. We also find a variety of Shil'nikov-type heteroclinic cycles including saddle-focus and bi-focal cycles as well as heteroclinic cycles between limit cycles. We also observe other instabilities due to perturbations outside the invariant subspaces which the heteroclinic cycles evolve in. In particular, we find something akin to a blow-out bifurcation [4] for the limit-cycle to limit-cycle heteroclinic cycle.

Using the results developed for the low-dimensional truncation, we prove a stability criterion for the heteroclinic cycles in the full PDE which we then use with numerical continuation to explore the existence and stability of the heteroclinic cycles in a higher-dimensional truncation.

The paper is outlined as follows. In Section 2 we set out the problem and discuss some of the properties of the observed heteroclinic cycles. This leads us to a minimum Fourier truncation of the CGL that still possesses heteroclinic cycles. A discussion of the numerical techniques is given in Section 2.3. In Section 3 we explore the heteroclinic cycles in the minimal truncation and analyse their persistence and existence in Section 3.1. An analytical