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Edge-primitive Cayley graphs on abelian groups and dihedral groups[☆]

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ABSTRACT

A graph is called *edge-primitive* if its automorphism group acts primitively on its edge set. In 1973, Weiss (1973) determined all edge-primitive graphs of valency three, and recently Guo et al. (2013,2015) classified edge-primitive graphs of valencies four and five. In this paper, we determine all edge-primitive Cayley graphs on abelian groups and dihedral groups.

1. Introduction

Throughout the paper, graphs are assumed to be finite, simple and undirected, with valency at least three. For a graph $\varGamma$, denote by $V\varGamma$, $E\varGamma$ and $\mathrm{Aut}\varGamma$ its vertex set, edge set and the full automorphism group, respectively.

A graph $\varGamma$ is called X-*edge-primitive* if $X \le \mathrm{Aut}\varGamma$ acts primitively on $E\varGamma$ (or equivalently, X acts transitively on $E\varGamma$ and its edge stabilizer is maximal in X). If X is transitive on $V\varGamma$, then $\varGamma$ is called X-*vertex-transitive*. For a positive integer s, an s-*arc* of $\varGamma$ is a sequence $\alpha_0, \alpha_1, \ldots, \alpha_s$ of $s+1$ vertices such that α_{i-1}, α_i are adjacent for $1 \le i \le s$ and $\alpha_{i-1} \ne \alpha_{i+1}$ for $1 \le i \le s-1$. Then $\varGamma$ is called (G, s)-*arc-transitive* if $G \le \mathrm{Aut}\varGamma$ is transitive on the set of s-arcs of $\varGamma$. A $(G, 1)$-arc-transitive graph is arc-transitive. By [9, Lemma 3.4], an edge-primitive regular graph is arc-transitive. If G is transitive on the set of s-arcs but not transitive on the set of $(s+1)$-arcs of $\varGamma$, then $\varGamma$ is called (G, s)-*transitive*; in particular, $\varGamma$ is simply called s-*transitive* while $G = \mathrm{Aut}\varGamma$.

Edge-primitive graphs contain many famous graphs. In 1973, Weiss [24] determined all edge-primitive cubic graphs: they are the complete bipartite graph $\mathrm{K}_{3,3}$, the Heawood graph of order 14, the Biggs–Smith cubic distance-transitive graph of order 102 and the Tutte–Coxeter graph (also known as Tutte's 8-cage or the Levi graph). Giudici and Li [9] systematically studied the O'Nan–Scott type of the automorphism groups of edge-primitive graphs, and determined the G-edge-primitive graphs while G is almost simple with socle a two dimensional projective group, and Li and Zhang [16] classified edge-primitive 4-arc-transitive graphs. More recently, Guo, Feng and Li [10,11] determined edge-primitive graphs of valencies four and five respectively, and the authors of this paper [19] determined edge-primitive graphs of prime valency with soluble edge stabilizers. Notice that the edge stabilizers of edge-primitive graphs with valency at most five are soluble. It seems infeasible to study general edge-primitive graphs for the valency bigger than five along this line because the edge stabilizer can be insoluble.

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A graph Γ is a *Cayley graph* on a group G if there is a subset $S \subseteq G \setminus \{1\}$, with $S = S^{-1} := \{s^{-1} \mid s \in S\}$, such that $V\Gamma = G$, and two vertices g and h are adjacent if and only if $hg^{-1} \in S$. This Cayley graph is denoted by $\mathsf{Cay}(G, S)$. It is well known that a graph Γ is isomorphic to a Cayley graph on a group G if and only if $\mathsf{Aut}\Gamma$ contains a subgroup which is isomorphic to G and acts regularly on $V\Gamma$ [1, Proposition 16.3]. For convenience, we sometimes call a Cayley graph on a dihedral group a *dihedrant*.

The family of Cayley graphs is one of the most important families of vertex-transitive graphs, and provide a rich source of many interesting graphs. It is thus natural to consider the following problem.

Problem 1. Characterize edge-primitive Cayley graphs.

The main purpose of this paper is to determine all edge-primitive Cayley graphs on abelian groups and dihedral groups. The notations used in this paper are standard. For example, for a positive integer n, we use $\mathbb{Z}_n$ and D_{2n} to denote the cyclic group of order n and the dihedral group of order $2n$ respectively. Given two groups N and H, denote by $N \times H$ the direct product of N and H, by $N.H$ an extension of N by H, and if such an extension is split, then we write $N : H$ instead of $N.H$.

The main results of this paper are the following two theorems, where $\mathcal{HD}(22)$ and $\overline{\mathcal{HD}}(22)$ denote the incidence and non-incidence graphs of the Hadamard design on 22 points (see Example 3.1 for detail); $\mathcal{PH}(d, q)$ and $\overline{\mathcal{PH}}(d, q)$ with $d \geq 3$ denote the point-hyperplane incidence and non-incidence graphs of $(d-1)$-dimension projective geometry $PG(d-1, q)$ (see Example 3.3 for detail). As usual, denote by K_n the complete graph of order n, and by $\mathsf{K}_{n,n}$ the complete bipartite graph of order $2n$. Notice that the corrected graphs with valency two are cycles.

Theorem 1.1. *Let Γ be a connected X-edge-primitive Cayley graph on an abelian group of valency at least three, with $X \leq \mathsf{Aut}\Gamma$. Then either*

(1) *X is primitive on $V\Gamma$, and $\Gamma = \mathsf{K}_n$ is 2-transitive with $n \geq 5$; or*
(2) *X is bi-primitive on $V\Gamma$, and $\Gamma = \mathsf{K}_{n,n}$ is 3-transitive with $n \geq 3$.*

Theorem 1.2. *Let Γ be a connected X-edge-primitive Cayley graph on a dihedral group of valency at least three, where $X \leq \mathsf{Aut}\Gamma$. Then either*

(1) *X is primitive on $V\Gamma$, and $\Gamma = \mathsf{K}_{2n}$ is 2-transitive with $n \geq 3$; or*
(2) *X is bi-primitive on $V\Gamma$, and one of the following holds.*
    (a) *$\Gamma = \mathsf{K}_{n,n}$ is 3-transitive with $n \geq 3$;*
    (b) *$\Gamma = \mathcal{HD}(22)$ or $\overline{\mathcal{HD}}(22)$ is 2-transitive, and $\mathsf{Aut}\Gamma \cong \mathsf{PGL}(2, 11)$;*
    (c) *$\Gamma = \mathcal{PH}(d, q)$ or $\overline{\mathcal{PH}}(d, q)$ with $d \geq 3$ and q a prime power, and $\mathsf{Aut}\Gamma \cong \mathsf{Aut}(\mathsf{PSL}(d, q))$. Further, the following hold:*
        (i) *$\mathcal{PH}(3, q)$ is 4-transitive;*
        (ii) *$\mathcal{PH}(d, q)$ is 2-transitive for $d \geq 4$;*
        (iii) *$\overline{\mathcal{PH}}(d, q)$ is 2-transitive for $d \geq 3$.*

Theorems 1.1 and 1.2 have the following corollary.

Corollary 1.3. *The connected edge-primitive Cayley graphs on abelian groups and dihedral groups are 2-arc-transitive.*

The known edge-primitive but not 2-arc-transitive graphs are rare (see [10,11,16,19,24]). This motivates the following problem.

Problem 2. Characterize edge-primitive graphs that are not 2-arc-transitive.

The structure of is paper is as follows. After this introductory section, we give some preliminary results regarding overgroups containing a regular abelian or dihedral subgroup in Section 2, and investigate examples appearing in Theorems 1.1 and 1.2 in Section 3. Then, by proving some technical lemmas in Section 4, we complete the proof of Theorems 1.1 and 1.2 in Sections 5 and 6 respectively.

2. *a*-groups and *d*-groups

Let X be a transitive permutation group on a set Ω. Then X is called *quasiprimitive* if each of its minimal normal subgroups is transitive on Ω, and X is called *bi-quasiprimitive* if each of its minimal normal subgroups has at most two orbits and there exists one which has exactly two orbits on Ω. A subset $B \subseteq \Omega$ is called a *block* of X if for each element $x \in X$, either $B^x = B$ or $B \cap B^x = \emptyset$. Obviously, single element sets and Ω are blocks, called *trivial blocks*. The other blocks (if any exist) of X are called *nontrivial blocks* or *imprimitive blocks*. The group X is called *primitive* if it has only trivial blocks, otherwise X is called *imprimitive*; and X is called *bi-primitive* if X has two blocks Δ_1 and Δ_2 on Ω such that $\Omega = \Delta_1 \cup \Delta_2$ and the block stabilizer $X_{\Delta_1} = X_{\Delta_2}$ is primitive on Δ_1 and Δ_2. A primitive permutation group is quasiprimitive.

For a Cayley graph on a group G, the right regular representation

$$\hat{G} := \{\hat{g} : x \to xg \mid g, x \in G\}$$