Computers and Geotechnics 102 (2018) 39–52

Contents lists available at ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

Research Paper

A novel approach for application of smoothed point interpolation methods to axisymmetric problems in poroelasticity

A. Tootoonchi[a], A. Khoshghalb[a,*], G.R. Liu[b]

[a] School of Civil and Environmental Engineering, UNSW Sydney, Sydney, NSW 2052, Australia
[b] Department of Aerospace Engineering and Engineering Mechanics, University of Cincinnati, Cincinnati, OH 45221, USA

ARTICLE INFO

Keywords:
Axisymmetric
Meshfree methods
Smoothing gradient operation
Poroelasticity
Smoothed point interpolation method

ABSTRACT

A smoothed point interpolation method (SPIM) for the numerical modelling of saturated porous media in axisymmetric conditions is proposed, aiming to overcome the singularity problem encountered when using SPIMs in axisymmetric settings. The singularity is circumvented in this study by decomposition of the property matrices of the system to sub-matrices with smoothed terms and non-smoothed terms. The salient feature of the proposed method is that it neither incurs additional computation nor compromises on the accuracy of the method. The proposed method is examined by numerical modelling of several benchmark axisymmetric problems, along with a set of convergence studies.

1. Introduction

Axisymmetric problems are of predominant importance in geotechnical engineering because of their relevance in many cases such as uniaxial and triaxial tests, pile installation, foundation loading, and deep tunnels. Many numerical techniques have been extended to axisymmetric conditions and often used for applications in geotechnical engineering [1–4]. In particular, finite element method (FEM) has been widely used as an important tool for numerical simulation of geotechnical engineering problems [e.g., 5–9].

Despite its wide-spread use, FEM has well-known inherent deficiencies that may result in poor accuracy or inefficiency of the numerical simulations. These deficiencies include strong reliance on the mesh quality, overestimation of the stiffness of the model, and poor performance when discontinuities are present. Attempts to address the difficulties associated with the use of FEM have led to development of meshfree methods (MMs). Since their inception, many MMs have been developed with different features and capabilities. For more detailed descriptions, interested readers are referred to [10–19].

Recently, a wide class of efficient MMs called smoothed point interpolation methods (SPIM) have been developed [20–22] by using the point interpolation methods (PIM) with the generalisation of the strain smoothing operation [23] referred to as generalised gradient smoothing operations [20]. In SPIMs, instead of using a compatible strain field, a smoothed strain field is constructed through a smoothing operation performed over smoothing domains. The use of smoothed strain field overcomes the problems in PIMs associated with discontinuity of the approximation field over the problem domain through elimination of the need for the derivatives of the shape functions. SPIMs are very flexible and can be formulated in many ways through different node selection schemes and different types of smoothing domains leading to different SPIMs [21,22,24,25]. The simplest SPIM is perhaps the cell-based SPIM (CSPIM) in which the cells of a triangular background mesh are used as the smoothing domains. To date, CSPIM has been successfully applied to plane strain problems in geotechnical engineering with results showing its superiority to similar methods (e.g. FEM) in terms of both efficiency and convergence rate [26].

Despite their excellent performance, application of SPIMs to axisymmetric problems has received little attention in the literature. This may be attributed to the difficulties associated with application of SPIMs in axisymmetric conditions due to the existence of Gauss points on the boundary of the smoothing domains on the axis of symmetry. Specifically, CSPIM cannot be directly extended to axisymmetric conditions due to the singularity problem that arises in the analysis. Wan et al. [27] has recently proposed a formulation for application of the smoothed finite element method (SFEM) in axisymmetric problems. To avoid the singularity problem associated with having Gauss points on the axis of symmetry, they have used an approximation based on employing the radius of the central point of the elements instead of radius of each Gauss point of interest. This approach may be reasonable for SFEM with simple shape functions, however, can be erroneous if directly extended to SPIMs because more supporting nodes, beyond the element limits, may be involved in construction of the shape functions in the latter. Furthermore, in the formulation suggested in [27], the

* Corresponding author.
E-mail address: Arman.khoshghalb@unsw.edu.au (A. Khoshghalb).

https://doi.org/10.1016/j.compgeo.2018.05.010
Received 28 March 2018; Received in revised form 14 May 2018; Accepted 24 May 2018

integration of the shape functions over the elements which appears in the smoothed strain-displacement matrix of an axisymmetric formulation is obtained analytically. This approach is, however, applicable only when simple shape functions are used, and in general, cannot be extended to SPIMs where complex shape functions (e.g., Radial PIM shape functions) may be adopted in the formulation.

In this paper, a novel approach is presented to extend the formulation of the CSPIM to axisymmetric conditions. The approach is general in nature and can also be applied to other SPIMs and SFEM to extend their application to axisymmetric coupled problems. The approach is based on separating the smoothed terms from the non-smoothed terms in the property matrixes of the discretised governing equations. The technique presented does not include any additional approximations or computational costs compared to the plain strain formulation [26], and therefore, does not adversely affect the accuracy or efficiency of the numerical procedure. The validity of the proposed method is verified through a suite of benchmark problems, ranging from theoretical to practical problems. The accuracy and convergence rate of the proposed formulation are studied by performing error analyses in terms of displacement and pore water pressure using a series of simulations with different background mesh densities. The presented method clearly manifests its superiority to FEM in the examples presented.

2. Governing equations

A two-phase porous medium consisting of a solid matrix and a saturating fluid is considered. In isothermal conditions, the linear momentum and mass balance equations for the medium, initially developed by Biot [28], can be expressed as:

$$\partial^T(\boldsymbol{\sigma}' - \gamma p_f \boldsymbol{I}) + \rho \boldsymbol{g} = 0 \quad \text{(Momentum balance equation)} \tag{1}$$

$$\gamma \boldsymbol{I}^T \partial \dot{\boldsymbol{u}} + \nabla^T \boldsymbol{v} + \alpha_f \dot{p}_f = 0 \quad \text{(continuity)} \tag{2}$$

with

$$a_f = n(c_f - c_s) + \gamma c_s, \quad \gamma = 1 - \frac{c_s}{c} \tag{3}$$

Here, $\boldsymbol{\sigma}'$ is the effective Cauchy stress tensor; p_f is the excess pore fluid pressure, denoting pressure in excess of the steady state fluid pressure; γ is the Biot's coefficient; $\boldsymbol{u}$ stands for the displacement field of the solid matrix; $\boldsymbol{v}$ denotes the superficial velocity (flux) field of the fluid phase, which is the average relative seepage velocity per unit cross section area; ρ is the buoyant density of the mixture in the saturating fluid (note that the formulation is based on the *excess* pore fluid pressure); and $\boldsymbol{g}$ represents the gravity vector. a_f is the apparent compressibility of the fluid, n indicates the porosity of the poroelastic medium, c_f is the compressibility of the fluid, c_s is the compressibility of solid grains and c is the drained compressibility of the solid skeleton. Voigt notation has been adopted in which second order symmetric tensors are written as column matrices, and fourth order symmetric tensors are written as square matrices [24]. The sign convention of continuum mechanics has been adopted, i.e., compressive stresses and strains are assumed negative. Pore fluid pressure (p_f) is, however, taken as positive in compression following the soil mechanics convention. Subscript f and s stand for fluid and solid, respectively. ∇ is the gradient operator, and ∂ is the differentiation matrix. For an axisymmetric setting, they are defined, respectively, as:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial z} \end{bmatrix}^T \tag{4-1}$$

$$\partial = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & \frac{\partial}{\partial z} & \frac{1}{r} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial r} & 0 \end{bmatrix}^T \tag{4-2}$$

where $\boldsymbol{x} = [r \quad z]$ is the polar coordinate system, with z being the axis of symmetry. $\boldsymbol{I}$ is the identity vector defined as $\boldsymbol{I} = [1 \quad 1 \quad 0 \quad 1]^T$ for an axisymmetric condition.

The superficial velocity is related to the excess pore fluid pressure by the generalised Darcy's law:

$$\boldsymbol{v} = \frac{\boldsymbol{k}_f}{\mu_f}(-\nabla p_f + \rho_f \boldsymbol{g}) \tag{5}$$

in which $\boldsymbol{k}_f$ is the tensor of intrinsic hydraulic permeability of the medium, ρ_f is the density of the fluid phase, and μ_f is the dynamic viscosity of the fluid phase.

A constitutive model is needed to relate the effective stress to the strain of the solid phase. For simplicity, small strains and elastic behaviour are assumed for the solid skeleton, resulting in the following relationships:

$$\dot{\boldsymbol{\sigma}}' = \boldsymbol{D}\dot{\boldsymbol{\varepsilon}} \tag{6}$$

where

$$\boldsymbol{\sigma}' = [\sigma_r' \quad \sigma_z' \quad \tau_{rz}' \quad \sigma_\theta']^T \tag{7-1}$$

$$\boldsymbol{\varepsilon} = [\varepsilon_r \quad \varepsilon_z \quad \gamma_{rz} \quad \varepsilon_\theta]^T = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_z}{\partial z} & \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} & \frac{u_r}{r} \end{bmatrix}^T \tag{7-2}$$

and $\boldsymbol{D}$ is the elastic constitutive matrix. u_r and u_z are displacement components, and $\boldsymbol{\varepsilon}$ is the strain of the solid skeleton. The overdot denotes the rate of change of the corresponding variable with respect to time.

It is assumed that the poroelastic medium fills a domain Ω with a boundary Γ. In a standard manner, the boundary is divided into regions where essential and natural boundary conditions for solid and fluid phases are specified, as follows:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \bar{\boldsymbol{u}}(t) \quad on\, \Gamma_u \tag{8}$$

$$\boldsymbol{n}^T\boldsymbol{\sigma}'(\boldsymbol{x}, t) = \bar{\boldsymbol{t}}(t) \quad on\, \Gamma_t \tag{9}$$

$$p_f(\boldsymbol{x}, t) = \bar{p}_f(t) \quad on\ \Gamma_p \tag{10}$$

$$\boldsymbol{n}^T\boldsymbol{v}(\boldsymbol{x}, t) = \bar{q}(t) \quad on\, \Gamma_q \tag{11}$$

where $\boldsymbol{n}$ is the outward unit normal vector, expressed in a matrix form as:

$$\boldsymbol{n} = \begin{bmatrix} n_r & 0 & n_z & 0 \\ 0 & n_z & n_r & 0 \end{bmatrix}^T \tag{12}$$

in which, n_r and n_z are the components of unit normal vectors in r and z directions, respectively. In a standard manner, we have:

$$\Gamma = \Gamma_u \cup \Gamma_t = \Gamma_p \cup \Gamma_q \tag{13}$$

$$\Gamma_u \cap \Gamma_t = \Gamma_p \cap \Gamma_q = \varnothing \tag{14}$$

Finally, initial conditions are expressed as

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \bar{\boldsymbol{u}}_0(\boldsymbol{x}) \tag{15}$$

$$p_f(\boldsymbol{x}, 0) = \bar{p}_{f0}(\boldsymbol{x}) \tag{16}$$

2.1. Variational form

The variational form of the governing equations is presented here. Two sub spaces of trial functions are employed as follows

$$S_u = \{\boldsymbol{u}: \Omega \to \mathbb{R}^2 \mid \boldsymbol{u} \in \boldsymbol{G}_h^1,\ \boldsymbol{u} = \bar{\boldsymbol{u}}\ on\ \Gamma_u\} \tag{17}$$

$$S_p = \{p_f: \Omega \to \mathbb{R} \mid p_f \in G_h^1,\ p_f = \bar{p}_f\ on\ \Gamma_p\} \tag{18}$$

where G^1 denotes a G space of degree one, and G_h^1 indicates a discretised subspace of G^1. G^1 is more accommodative than the well-known H^1 (Sobolev space of degree one) in a sense that in G^1 only the function itself is required to be square integrable, as opposed to H^1 which requires the first gradient of the function to be also square integrable