International Journal of Approximate Reasoning 102 (2018) 108–122

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Type-2 fuzzy implications and fuzzy-valued approximation reasoning

Chun Yong Wang [a,*], Lijuan Wan [b]

[a] School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, PR China
[b] School of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao 266061, PR China

A R T I C L E I N F O

Article history:
Received 27 May 2018
Received in revised form 12 August 2018
Accepted 17 August 2018
Available online 23 August 2018

Keywords:
Type-2 fuzzy sets
Extended fuzzy implications
Fuzzy values
Fuzzy-valued fuzzy implications
Fuzzy-valued approximate reasoning

A B S T R A C T

In this paper, the quasi-distributivity laws of type-2 fuzzy implications with respect to extended supremum and extended infimum are investigated on different subalgebras of fuzzy truth values. Moreover, the properties of fuzzy-valued fuzzy implications are further discussed. Especially, new fuzzy-valued operations are obtained from a fuzzy-valued fuzzy implication induced by a fuzzy implication, which is right-continuous with respect to the second argument. As an application of fuzzy-valued fuzzy implications, fuzzy-valued approximate reasoning is further studied and a numerical example is given.

1. Introduction

Type-2 fuzzy sets were proposed by Zadeh [44] as a generalization of ordinary fuzzy sets [42]. Hence type-2 fuzzy sets provide more design degree of freedom in practice. Since both ordinary fuzzy sets and interval-valued fuzzy sets are special cases of type-2 fuzzy sets, it is pointed out in [2] that type-2 fuzzy sets are very useful in circumstances where there is need to handle more uncertainty than it is possible using ordinary fuzzy sets or interval-valued fuzzy sets. Thus, type-2 fuzzy sets have been applied in many areas, such as prediction [4,12,30], recognition [31], clustering [26,28,45], optimization [34,41], decision making [1,32] and control [5,27].

Karnik et al. [22] constructed type-2 fuzzy logic systems from the view of type reductions [18,35] and centroids [23,33]. Type-2 fuzzy logic systems were investigated in [9,17,29,37,40]. Especially, many researchers have studied type-2 fuzzy implications [13,14,25,38,39,46]. For example, Wang and Hu [38] proposed a fuzzy-valued fuzzy implication generated from a fuzzy-valued t-norm induced by a left-continuous t-norm. Li [25] discussed the residual operators of type-2 t-norms with respect to the partial order $\sqsubseteq$ induced by extended minimum, which were further investigated in [46]. Generalized extended fuzzy implications were investigated in accordance with the generalized extension principle in [39], where neither t-norms nor fuzzy implications are necessarily continuous. Moreover, fuzzy-valued fuzzy implications induced by arbitrary fuzzy implications and t-norms were proposed on the algebra of fuzzy values regardless of the continuity of either fuzzy implications or t-norms. However, the quasi-distributivity laws of type-2 fuzzy implications with respect to extended supremum and extended infimum are not fully discussed in [39], where Wang and Hu only investigated the quasi-distributivity laws of extended fuzzy implications with respect to extended maximum and extended minimum. These suggest that extended fuzzy implications should be studied more thoroughly in this context. Hence, we discuss the quasi-distributivity laws of

* Corresponding author.
E-mail addresses: chunyong_wang@163.com (C.Y. Wang), wanjuan600@163.com (L. Wan).

https://doi.org/10.1016/j.ijar.2018.08.004
0888-613X/

extended fuzzy implications with respect to extended supremum and extended infimum on different subalgebras of fuzzy truth values, such as convex and normal ones. On this basis, we further investigate quasi-distributivity laws of fuzzy-valued fuzzy implications induced by arbitrary fuzzy implications with respect to extended supremum and extended infimum. Moreover, we propose new fuzzy-valued operations based on a fuzzy-valued fuzzy implication to enrich the operations on fuzzy values, where fuzzy implication is right-continuous with respect to the second argument.

One of the best known application areas of fuzzy logic is approximate reasoning, referring to methods and methodologies that enable reasoning with imprecise inputs to obtain meaningful outputs [10]. Chen and Kawase [8] introduced fuzzy-valued approximate reasoning from the perspective of interval-valued approximate reasoning [6]. However, the work of Chen and Kawase requires the continuity of both t-norms and fuzzy implications. In this paper, fuzzy-valued approximate reasoning is further investigated with arbitrary t-norms and fuzzy implications, where neither t-norms nor fuzzy implications are necessarily continuous.

The content of this paper is organized as follows. In Section 2, we recall some fundamental concepts and related properties about extended fuzzy implications. In Section 3, we discuss the quasi-distributivity laws of extended fuzzy implications with respect to extended supremum and extended infimum on the algebra of convex (resp. normal) fuzzy truth values. Section 4 studies the quasi-distributivity laws of fuzzy-valued fuzzy implications induced by arbitrary fuzzy implications with respect to extended supremum and extended infimum. Moreover, new fuzzy-valued operations are proposed to enrich the operations on fuzzy values. In Section 5, as an application of fuzzy-valued fuzzy implications, we further study fuzzy-valued approximate reasoning with arbitrary t-norms and fuzzy implications. In the final section, our research is concluded.

2. Preliminaries

Let X and Y be nonempty sets, which are referred to be universes. The family of all mappings from X to Y is denoted as $\mathscr{M}(X,Y)$. The symbol I always denotes the unit interval $[0,1]$ in this paper. Especially, if $Y=I$, then $\mathscr{M}(X,Y)$ is denoted as $\mathscr{M}(X)$ for short.

A fuzzy set A is a mapping from X to I, i.e., $A\in\mathscr{M}(X)$. Especially, a fuzzy set A is called a *fuzzy truth value*, if $A\in\mathscr{M}(I)$. Two crisp sets $\emptyset$ and X are special elements in $\mathscr{M}(X)$, with $\emptyset(x)=0$ and $X(x)=1$ for all $x\in X$, respectively. The order relation on fuzzy sets is defined as $A\subseteq B \Leftrightarrow A(x)\leqslant B(x)$ for all $x\in X$. Let $a,b\in I$ and $a\leqslant b$. Then a special fuzzy truth value A is denoted as $[a,b]$, if

$$A(x)=\begin{cases}1, & \text{if } a\leqslant x\leqslant b;\\ 0, & \text{otherwise.}\end{cases}$$

Especially, if $a=b$, then $[a,b]$ is denoted as $\overline{a}$ for short. An α-cut set of fuzzy set A on X is $A_\alpha=\{x\in X|A(x)\geqslant\alpha\}$ for all $\alpha\in I$. A strong α-cut set of fuzzy set A is $A_{\overline{\alpha}}=\{x\in X|A(x)>\alpha\}$ for all $\alpha\in I$. The least upper bound of A is denoted by $A^{\sup}$, i.e., $A^{\sup}=\sup_{x\in X}A(x)$.

A binary operation $\top: I\times I\to I$ (resp. $\bot: I\times I\to I$) is called a *t-norm* (resp. *t-conorm*) on I if it is commutative, associative, increasing in each argument and has a unit element 1 (resp. 0).

The associativity of t-norm $\top$ allows us to define the power notation $x_\top^{[n]}$ with $n\in\mathbb{N}_0$, by induction:

$$x_\top^{[n]}=\begin{cases}1, & \text{if } n=0,\\ x\top x_\top^{[n-1]}, & \text{otherwise.}\end{cases}$$

Some subclasses of t-norms are recalled as follows:

Definition 2.1 *([3,24]).* A t-norm $\top$ is said to be

(1) *continuous*, if it is continuous in each argument;
(2) *left-continuous*, if it is left-continuous in each argument;
(3) *nilpotent*, if it is continuous and each $x\in(0,1)$ is a nilpotent element of $\top$, i.e., if there exists an $n\in\mathbb{N}$ such that $x_\top^{[n]}=0$.

A binary operation $\lhd: I\times I\to I$ is called a *fuzzy implication* on I if it satisfies the boundary conditions according to the Boolean implication, and is decreasing in the first argument and increasing in the second argument.

The most important of additional properties of fuzzy implications are presented as follows.

Definition 2.2 *([3]).* A fuzzy implication $\lhd$ is said to satisfy

(1) the *left neutrality property* ((NP), for short), if $1\lhd y=y$ for all $y\in I$;
(2) the *exchange principle* ((EP), for short), if $x\lhd(y\lhd z)=y\lhd(x\lhd z)$ for all $x,y,z\in I$;
(3) the *ordering property* ((OP), for short), if $x\lhd y=1\Leftrightarrow x\leqslant y$, for all $x,y\in I$.