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Polynomial-time algorithm for weighted efficient domination problem on diameter three planar graphs

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ABSTRACT

A set D of vertices in a graph G is an *efficient dominating* set (e.d.s. for short) of G if D is an independent set and every vertex not in D is adjacent to exactly one vertex in D. The efficient domination (ED) problem asks for the existence of an e.d.s. in G. The minimum weighted efficient domination problem (MIN-WED for short) is the problem of finding an e.d.s. of minimum weight in a given vertex-weighted graph. Brandstädt, Fičur, Leitert and Milanič (2015) [3] stated the running times of the fastest known polynomial-time algorithms for the MIN-WED problem on some graphs classes by using a Hasse diagram. In this paper, we update this Hasse diagram by showing that, while for every integer d such that $d = 3k$ or $d = 3k + 2$, where $k \geq 1$, the ED problem remains NP-complete for graphs of diameter d, the weighted version of the problem is solvable in time $\mathcal{O}(|V(G)| + |E(G)|)$ in the class of diameter three bipartite graphs and in time $\mathcal{O}(|V(G)|^5)$ in the class of diameter three planar graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph. The *distance* between two vertices u and v of a connected graph G, denoted $d_G(u, v)$, is the number of edges in a shortest path from u to v. The *eccentricity* $e_G(u)$ of a vertex u, of a connected graph G, is $max\{d_G(u, v) | v \in V(G)\}$. The *radius* of a connected graph G, *rad(G)*, is the minimum eccentricity among the vertices of G, while the *diameter* of G, *diam(G)*, is the maximum eccentricity.

A *neighbor* of a vertex v in the graph G is a vertex adjacent to v. The *open neighborhood* of v, denoted $N_G(v)$, is the set of all neighbors of v. The *closed neighborhood of v* is $N_G[v] = \{v\} \cup N_G(v)$. For a subset S of vertices of G, the closed neighborhood of S, denoted $N_G[S]$, is $\bigcup_{v \in S} N_G[v]$. For $k \geq 1$, we use the notation $[k] = \{1, \dots, k\}$. Recall that the *path graph* and the *complete graph* on n vertices are denoted by P_n and K_n, respectively.

A set D of vertices in a graph G is an *efficient dominating* set (e.d.s. for short) of G if D is an independent set and every vertex not in D is adjacent to exactly one vertex in D. Note that not every graph has an e.d.s. The efficient domination (ED) problem asks for the existence of an e.d.s. in G. The *minimum weighted efficient domination* problem (MIN-WED for short) is the problem of finding an efficient dominating set of minimum weight in a given vertex-weighted graph. The notion of efficient domination was introduced by Biggs [2] under the name *perfect code*. Later, Bange, Barkauskas and Slater [1] showed that the ED problem is NP-complete. Furthermore, they showed that the ED problem is solvable in polynomial time on trees. Many papers have studied the complexity of the ED problem and also the weighted version of the ED problem on special graph classes. Brandstädt et al. [3] presented an interesting Hasse diagram of the poset of most of such graph classes. For each class, they stated the complexity of weighted version of the ED problem. In par-

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ticular, the ED problem is known to be NP-complete [5] for $2P_3$-free graphs and thus for P_7-free graphs and solvable in polynomial time for weighted version of the ED problem on P_5-free and also P_6-free graphs. Brandstädt, Milanič and Nevries [4] presented polynomial-time algorithms for weighted version of the ED problem for various subclasses of $2P_3$-free graphs as well as of P_7-free graphs, including $(P_2 + P_4)$-free graphs and other classes. Moreover, they showed in [4] that the ED problem is NP-complete for planar bipartite graphs with maximum degree three. Recently, Brandstädt and Mosca [5] found an $\mathcal{O}(|V(G)|^5|E(G)|)$ time solution for the weighted version of the ED problem on P_6-free graphs which improved the time bound of (at least) $\mathcal{O}(|V(G)|^{576})$ due to Lokshtanov et al. [6]. Moreover, they showed in [5] that the weighted version of the ED problem is solvable in linear time for P_5-free graphs. According to the mentioned results, all open questions regarding the complexity of the ED problem on P_k-free graphs were answered.

Now, we focus on similar questions for the complexity of the ED problem on graphs of diameter k. Note that for every $k \geq 2$, every connected P_k-free graph is of diameter at most $k-2$. In particular, this implies that the ED problem on graphs of diameter at most three, resp. four, is at least as difficult as on P_5-free, resp. P_6-free graphs. It is easy to see that a graph G of diameter two has an e.d.s. if and only if $rad(G) = 1$. Therefore, the ED problem is solvable in linear time for graphs of diameter two. To the best of our knowledge, the question for the complexity of the ED problem on graphs of diameter k, where $k \geq 3$, remained unsolved. In this paper, we prove that for every integer d such that $d = 3k$ or $d = 3k+2$, where $k \geq 1$, the ED problem remains NP-complete for graphs of diameter d. Furthermore, we show that the MIN-WED problem is solvable in time $\mathcal{O}(|V(G)|^5)$ in the class of diameter three planar graphs. Finally, we present a simple algorithm for the MIN-WED problem in time $\mathcal{O}(|V(G)| + |E(G)|)$ for the class of diameter three bipartite graphs.

The main results of this paper are the following:

1. In Section 2, by using a new graph construction, which we call a *semi-Mycielski graph*, we prove that the ED problem is NP-complete on diameter three graphs. Furthermore, by using the reduction establishing NP-completeness of the ED problem for chordal graphs due to Yen and Lee [7, Theorem 2], we show that the ED problem is NP-complete on diameter five graphs.
2. In Section 3, we study the complexity status of the ED problem on graphs with larger diameter. We prove that for every integer d such that $d = 3k$ or $d = 3k+2$, where $k \geq 1$, the ED problem remains NP-complete for graphs of diameter d.
3. In Section 4, we show that if a planar graph G of diameter three has an e.d.s., then G has an e.d.s. of size at most 4. Thus, we obtain an $\mathcal{O}(|V(G)|^5)$ time algorithm for solving the MIN-WED problem on diameter three planar graphs. Finally, we present an algorithm in time $\mathcal{O}(|V(G)| + |E(G)|)$ for solving the MIN-WED problem on diameter three bipartite graphs.

2. Graphs with small diameter

Our aim in this section is to prove that the ED problem for the class of diameter three graphs and also for the class of diameter five graphs is NP-complete.

Theorem 1. *The ED problem is NP-complete on diameter three graphs.*

In order to prove Theorem 1, we define the semi-Mycielski graph of a graph G. Let G be a graph with vertex set $V(G)$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *semi-Mycielski graph* of G, denoted $R(G)$, is the graph obtained from G by taking a vertex-disjoint copy of a complete graph K_n with vertex set $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and join u_i to $N_G[v_i]$ for every $i \in [n]$.

The following lemma proves that if G has diameter at least three, then the vertices of $R(G)$ have eccentricity 2 or 3. Moreover, $R(G)$ has diameter three.

Lemma 2. *Let G be a graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $diam(G) \geq 3$. Then the graph $R(G)$ with the vertex set $\{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ has the following properties:*

(i) *For all $i \in [n]$, we have $e_{R(G)}(v_i) \leq 3$.*
(ii) *For all $i \in [n]$, we have $e_{R(G)}(u_i) = 2$.*
(iii) *There are at least two distinct vertices v_s and v_t in $V(G)$ such that $e_{R(G)}(v_t) = e_{R(G)}(v_s) = 3$.*
(iv) *$diam(R(G)) = 3$.*

Proof. Let $v_j \in V(G)$ for some $j \in [n]$. Consider an arbitrary vertex $v_i \in V(G)$ for some $i \in [n]$. If $d_G(v_j, v_i) \geq 4$, then in the graph $R(G)$, there is a path, v_j, u_j, u_i, v_i, of length three from v_j to v_i. Thus, $d_{R(G)}(v_j, v_i) \leq 3$. If $d_G(v_j, v_i) \leq 3$, then it is easy to see that $d_{R(G)}(v_j, v_i) = d_G(v_j, v_i) \leq 3$. Now consider an arbitrary vertex $u_i \in \{u_1, u_2, \ldots, u_n\}$. If $i = j$, then $d_{R(G)}(v_j, u_i) = 1$. Otherwise, if $i \neq j$ and $v_i v_j \notin E(R(G))$, then $d_{R(G)}(v_j, u_i) = 2$, since there is a path v_j, u_j, u_i of length two from v_j to u_i. If $i \neq j$ and $v_i v_j \in E(R(G))$, then by definition of $R(G)$, $v_j u_i \in E(R(G))$. Hence, $d_{R(G)}(v_j, u_i) = 1$. Thus, for an arbitrary vertex x of $V(R(G))$, we have $d_{R(G)}(v_j, x) \leq 3$. It follows that $e_{R(G)}(v_j) \leq 3$, establishing (i). The proof of (ii) is similar to (i). Thus, $diam(R(G)) \leq 3$. Since $diam(G) \geq 3$, there are at least two distinct vertices v_s and v_t in $V(G)$ such that $e_G(v_t) = e_G(v_s) \geq 3$. It is easy to see that $e_{R(G)}(v_t) = e_{R(G)}(v_s) = 3$. Therefore, $diam(R(G)) = 3$. This completes the proof of Lemma 2. □

The vertices of eccentricity two play an important role in the graph $R(G)$. The following lemma shows that such vertices are not in any e.d.s. of $R(G)$.

Lemma 3. *If u is an arbitrary vertex of a graph G such that $e_G(u) = 2$, then for every e.d.s. D of G, $u \notin D$.*

Proof. Suppose, to the contrary, that $u \in D$. Since $e_G(u) = 2$, we have $N_G[N_G[u]] = V(G)$. By definition of e.d.s., we note that $N_G[N_G[u]] \cap D = \{u\}$. Thus, $D = \{u\}$.