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Inclusion problem in second gradient elasticity

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ABSTRACT

The Green's function and Eshelby tensors of an infinite linear isotropic second gradient continuum are derived for an inclusion of arbitrary shape. Particularly for spherical, cylindrical and ellipsoidal inclusions, Eshelby tensors and their volume averages are obtained in an analytical form. It is found that the Eshelby tensors are not uniform inside the inclusion even for a spherical inclusion, and their variations depend on the two characteristic lengths of second gradient theory. When size of inclusion is large enough compared to the characteristic lengths, the Eshelby tensor of the second gradient medium is reduced to the classical one, as expected. It is also demonstrated that the existing Green's functions and Eshelby tensors of couple stress theory, Aifantis, Kleinert and Wei–Hutchinson special strain gradient theories could be recovered as special cases. This work paves the way for constructing micromechanical method to predict size effect of composite materials, as shown for the effective modulus of particulate composite derived with the proposed theory.

1. Introduction

Microcontinuum theory (Eringen, 1999) is considered as an efficient tool to characterize overall mechanics response for microstructured materials (Buechner & Lakes, 2003; Chen, Lee, & Eskandarian, 2004; Eringen, 1999). This high order theory incorporates the micro-deformation of microstructure inside of a material point in addition to translation of its inertia center. Depending on choice of micro-deformation mode, different simplified microcontinuum theories are proposed. For example, micromorphic theory (Eringen, 1999; Forest & Sievert, 2003) assumes an arbitrary constant micro-deformation, it is the most general first grade microcontinuum theory. More specified theory can be developed by further assuming this constant micro-deformation, for example an independent rigid rotation, this gives micropolar theory (Eringen, 1968). For small deformation and slow motion assumption (Shaat & Abdelkefi, 2016), the micromorphic theory is consistent with microstructure theory of Mindlin (1964). Second gradient theory (Germain, 1973), which is also called strain gradient theory, is a special case of microstructure theory (Mindlin, 1964) by specifying the micro-deformation to be macro-displacement-gradient. Therefore the theory has only the macro-displacement as degree of freedom and is easy to be implemented. There are several well-known simplified versions of second gradient theory, such as the couple stress theory (Koiter, 1964; Mindlin & Tiersten, 1962; Toupin, 1962) corresponding to letting the micro-deformation gradient be only rotation gradient, Kleinert strain gradient theory (Kleinert, 1989) corresponding to considering the gradient of volumetric strain in addition to rotation

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gradient as the micro-deformation gradient, etc. Some other models, like Aifantis (Altan & Aifantis, 1997; Gao & Park, 2007) and Wei & Hutchinson's strain gradient elasticity theory (Song, Liu, Ma, Liang, & Wei, 2014; Wei, 2006; Wei & Hutchinson, 1997), use strain gradient or second gradient of displacement as the micro-deformation gradient and define different constitutive relations. The objective of developing these high order continuum theories is to characterize the size effect well observed when size of structure is decreasing to micro or nano scale (Fleck, Muller, Ashby, & Hutchinson, 1994; Kouzeli & Mortensen, 2002), the microstructure comes into play in this case which is unable to be accounted for by Cauchy continuum theory without microstructure (Hu, Liu, & Lu, 2005).

In order to predict the size effect manifested in composites materials, proper homogenization method should be established. Inclusion problem is an essential step to build micromechanical models, in which Eshelby tensor is a key factor. Eshelby tensors in some microcontinuum models have already been obtained. For example, Eshelby tensors of spherical, cylindrical inclusions (Cheng & He, 1995, 1997) and ellipsoidal inclusion (Ma & Hu, 2006) are derived for micropolar medium and Aifantis strain gradient medium (Gao & Ma, 2009, 2010; Ma & Gao, 2010; Zheng & Zhao, 2004) derived Eshelby tensor for spherical inclusion in couple stress medium. Zhang and Sharma (2005) examined Eshelby tensor of Kleinert's strain gradient theory. Unlike the classical Eshelby tensor, these Eshelby tensors are not uniform inside the inclusion domain. But based on the average of these Eshelby tensors, an average equivalent inclusion method could be established for composites (Liu & Hu, 2005; Ma & Gao, 2014; Sharma & Dasgupta, 2002; Xun, Hu, & Huang, 2004), and it can be used to predict the size-dependence of inclusion on overall elastoplastic property of composites.

As discussed above, the second gradient theory is a more general high order theory and easy to use, it may offer an alternative and flexible tool to establish homogenization method for composites. However inclusion problem of a general isotropic second gradient medium has not been addressed yet, which is the objective of this manuscript. The Green's function and Eshelby tensor will be derived in analytical form, and their interconnections with the existing strain gradient theories, couple stress, Aifantis, Kleinert, and Wei & Hutchinson's models will be demonstrated. Finally effective modulus of a particulate composite will be presented to illustrate the capacity to predict size effect. The manuscript is arranged as follows, in Section 2, Green's function of a general isotropic second gradient medium will be derived. The inclusion problem will be examined in Section 3, and its connection with different strain gradient theories will be discussed in Section 4. Some examples will be presented in Section 5 to illustrate characteristic of derived Eshelby tensor. In Section 6, effective modulus of a composite with spherical inclusion will be given to characterize size effect of mechanical behavior. Finally some conclusions are presented.

2. Green's function

For a linear isotropic second gradient continuum, the governing equations are (Mindlin, 1964):

geometrical relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ji}, \quad \eta_{ijk} = \nabla\nabla\mathbf{u} = u_{k,ij} = \eta_{jik} \tag{2.1}$$

equilibrium equation:

$$\sigma_{ik,i} - \tau_{ijk,ij} + f_k = 0 \tag{2.2}$$

constitutive equations:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} \quad \tau_{ijk} = \frac{\partial W}{\partial \eta_{ijk}} = T_{ijklmn}\eta_{lmn} \tag{2.3}$$

where u_i is displacement vector, ε_{ij} and η_{ijk} are strain and strain gradient tensors, σ_{ij} and τ_{ijk} are stress and high-order stress tensors. f_k is body force vector. W is strain energy density function, its expression is (Mindlin, 1965):

$$W = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1\eta_{ijj}\eta_{ikk} + a_2\eta_{iik}\eta_{kjj} + a_3\eta_{iik}\eta_{jjk} + a_4\eta_{ijk}\eta_{ijk} + a_5\eta_{ijk}\eta_{kji} \tag{2.4}$$

where λ, μ are Lame constants, a_1, a_2, a_3, a_4, a_5 are additional constants introduced in second gradient theory. C_{ijkl} and T_{ijklmn} are elasticity tensors of second gradient materials, their expressions are:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \tag{2.5}$$

$$\begin{aligned} T_{ijklmn} = {} & \frac{a_1}{2}\left(\delta_{il}\delta_{jk}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jl}\delta_{mn}\right) \\ & + \frac{a_2}{2}\left(\delta_{ij}\delta_{lk}\delta_{mn} + \delta_{ij}\delta_{mk}\delta_{ln} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{ik}\delta_{jn}\delta_{ml}\right) + 2a_3\delta_{ij}\delta_{kn}\delta_{lm} \\ & + a_4\left(\delta_{im}\delta_{jl}\delta_{kn} + \delta_{il}\delta_{jm}\delta_{kn}\right) + \frac{a_5}{2}\left(\delta_{in}\delta_{jm}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{lk} + \delta_{il}\delta_{jn}\delta_{mk}\right) \end{aligned} \tag{2.6}$$

δ_{ij} is Kronecker delta.