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Using a Factored Dual in Augmented Lagrangian Methods for Semidefinite Programming

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Abstract

In the context of augmented Lagrangian approaches for solving semidefinite programming problems, we investigate the possibility of eliminating the positive semidefinite constraint on the dual matrix by employing a factorization. Hints on how to deal with the resulting unconstrained maximization of the augmented Lagrangian are given. We further use the approximate maximum of the augmented Lagrangian with the aim of improving the convergence rate of alternating direction augmented Lagrangian frameworks. Numerical results are reported, showing the benefits of the approach.

Keywords: Semidefinite Programming, Alternating Direction Augmented Lagrangian method, theta function
2010 MSC: 90C22, 90C30, 90C06

1. Introduction

Semidefinite Programs (SDP) can be solved in polynomial time to some fixed prescribed precision, but the computational effort grows both with the number m of constraints and with the order n of the underlying space of symmetric matrices. Interior point methods to solve SDP become impractical both in terms of computation time and memory requirements, once $m \geq 10^4$. Several algorithmic alternatives have been introduced in the literature, including some based on augmented Lagrangian approaches [1, 2, 3, 4, 5]. It is the purpose of this paper to elaborate on the alternating direction augmented Lagrangian (ADAL) algorithms proposed in [3, 4] by introducing computational refinements. The key idea will be to eliminate the positive semidefinite constraint on the dual matrix by employing a factorization, so that the maximization of the augmented Lagrangian function with respect to the dual variables can be performed in an unconstrained fashion.

In the remainder of this section we give the problem formulation and state our notations. In Section 2 a description of the ADAL methods for solving semidefinite programs is given. Details on how we maximize the augmented Lagrangian, after the factorization of the dual matrix, are given in Section 3. In Section 4, we outline our new algorithm `DADAL`: an additional update of the dual variable within one iteration of the ADAL method is used as improvement step. The convergence of `DADAL` easily follows by the analysis done in [5], that looks at ADAL as a fixed point method. We give insights on how `DADAL` can improve the convergence rate of ADAL in Section 4.1. Section 5 shows numerical results and Section 6 concludes.

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1.1. Problem Formulation and Notations

Let $\mathcal{S}_n$ be the set of n-by-n symmetric matrices and $\mathcal{S}_n^+ \subset \mathcal{S}_n$ be the set of positive semidefinite matrices. Denoting by $\langle X, Y\rangle = \text{trace}(XY)$ the standard inner product in $\mathcal{S}_n$, we write the standard primal-dual pair of SDP problems as follows:

$$\begin{array}{ll} \min & \langle C, X\rangle \\ \text{s.t.} & \mathcal{A}X = b, \\ & X \in \mathcal{S}_n^+ \end{array} \tag{1}$$

and

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & C - \mathcal{A}^\top y = Z \\ & Z \in \mathcal{S}_n^+, \end{array} \tag{2}$$

where $C \in \mathcal{S}_n$, $b \in \mathbb{R}^m$, $\mathcal{A} : \mathcal{S}_n \to \mathbb{R}^m$ is the linear operator $(\mathcal{A}X)_i = \langle A_i, X\rangle$ with $A_i \in \mathcal{S}_n$, $i = 1, \ldots, m$ and $\mathcal{A}^\top : \mathbb{R}^m \to \mathcal{S}_n$ is its adjoint, $\mathcal{A}^\top y = \sum_i y_i A_i$.

We assume that both problems have strictly feasible points, so that strong duality holds. Under this assumption, (X, y, Z) is optimal if and only if

$$X \in \mathcal{S}_n^+, \quad \mathcal{A}X = b, \quad Z \in \mathcal{S}_n^+, \quad C - \mathcal{A}^\top y = Z, \quad ZX = 0. \tag{3}$$

We further assume that matrix A has full rank.

Let $v \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m\times n}$. In the following, we denote by $\text{vec}(M)$ the mn-dimensional vector formed by stacking the columns of M on top of each other (vec^{-1} is the inverse operation). We also denote by $\text{Diag}(v)$ the diagonal matrix having v in the diagonal. With e_i we denote the i-th vector of the standard basis in $\mathbb{R}^n$. Whenever a norm is used, we consider the Frobenius norm in case of matrices and the Euclidean norm in case of vectors. We denote the projection of some symmetric matrix S onto the positive semidefinite cone by $(S)_+$ and its projection onto the negative semidefinite cone by $(S)_-$.