Statistics and Probability Letters 143 (2018) 43–46

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Brownian bricklayer: A random space-filling curve

Noah Forman

Department of Mathematics, University of Washington, Seattle, WA 98118, USA

ARTICLE INFO

Article history:
Received 24 November 2017
Accepted 13 July 2018
Available online 6 August 2018

MSC:
60J55
60G17

Keywords:
Local times
Space-filling curve
Brownian motion

ABSTRACT

Let $(B(t),\ t \geq 0)$ denote the standard, one-dimensional Wiener process and $(\ell(y,t);\ y \in \mathbb{R},\ t \geq 0)$ its local time at level y up to time t. Then $\big((B(t),\ \ell(B(t),t)),\ t \geq 0\big)$ is a random, continuous path that fills the upper half-plane, covering one unit of area per unit time.

There is a tradition of classical, so-called "pathological" properties for functions arising naturally in the study of Brownian motion. Here are few examples of this. One-dimensional Brownian motion is a continuous, nowhere differentiable function. Its zero set in a bounded interval, $\{t \in [0,1] : B(t) = 0\}$ is homeomorphic to the Cantor set. Despite this set having zero Lebesgue measure, there is a way of quantifying how much time the Brownian path spends at (or more precisely, near) zero (or any other value $y \in \mathbb{R}$), called the *local time*:

$$\ell(y,t) := \lim_{\epsilon \to 0} (2\epsilon)^{-1} \mathrm{Leb}\big\{u \in [0,t] : |B(u) - y| < \epsilon\big\}. \tag{1}$$

The process $(\ell(0,t),\ t \geq 0)$ is then a randomized variant of the Cantor function, also known as the Devil's staircase: it is continuous everywhere, has derivative zero almost everywhere, and yet it increases without bound. See Mörters and Peres (2010, Chapter 6) for more background on Brownian local times.

The existence of space-filling curves was first proved by Peano, who gave a deterministic, recursive construction, which was later simplified by Mandelbrot (1977). This note seeks to bring attention to another pathological example arising from Brownian motion, which has been neglected in the literature: that of a random space-filling curve arising naturally from Brownian local times.

The discrete random bricklayer

Imagine a worker building a wall by stacking blocks vertically, in side-by-side columns. The worker starts by setting a single block in front of themselves. They then flip a coin, stepping to the right if they get heads or to the left if tails before placing the next block. They repeat this process ad infinitum, flipping, moving, and stacking blocks. See Fig. 1.

We label sites by integers, with the worker's initial position labeled '0'. The worker's left-to-right motion is the *simple random walk on* $\mathbb{Z}$. The number of blocks stacked at site j after the first n steps is called the *occupation time* of the walk at j up to time n.

The simple random walk on $\mathbb{Z}$ is transitive and recurrent (Durrett, 2010), meaning that it visits every site infinitely many times. Thus, the wall ultimately grows infinitely high at every site. Now suppose the blocks are labeled by the time at which

E-mail address: nforman@uw.edu.

https://doi.org/10.1016/j.spl.2018.07.010
0167-7152/

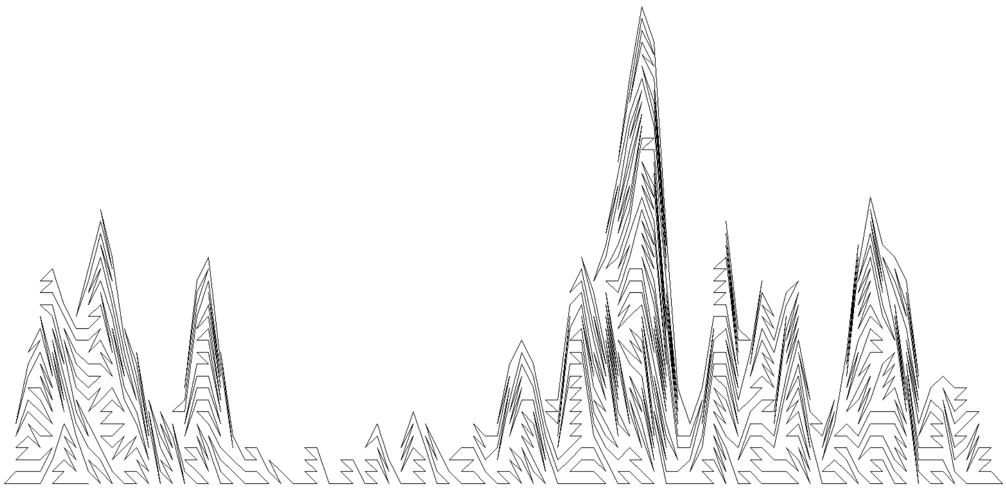

Fig. 1. A simulated trajectory of the discrete bricklayer, run for 1000 steps.

they were placed and we run a string across the front of the infinite wall, connecting consecutively numbered blocks. It is possible that adjacent columns can be at significantly different heights, so the string may run nearly vertical. But this will be uncommon, with the string usually taking smaller vertical steps, since the worker will have visited adjacent sites similar numbers of times (Csörgő and Révész, 1985, Theorem 1).

The Brownian bricklayer

For each $n \geq 0, j \in \mathbb{Z}$, let $S(n)$ denote the worker's position after n steps and $L(j, n)$ the height of the column at site j at this time. Donsker's theorem states that simple random walk converges in distribution, as a process, in a *scaling limit* to standard, one-dimensional Brownian motion:

$$\left(n^{-1/2}S\lceil nt\rceil,\ t \geq 0\right) \stackrel{d}{\to} (B(t),\ t \geq 0). \tag{2}$$

Knight (1963) strengthened this result, showing the joint convergence

$$\begin{aligned}&\left(\left(n^{-1/2}S\left(\lceil nt\rceil\right),\ t \geq 0\right),\ \left(n^{-1}L\left(\lceil ny\rceil, \lceil nt\rceil\right);\ t \geq 0,\ y \in \mathbb{R}\right)\right)\\ &\quad \stackrel{d}{\to} \left((B(t),\ t \geq 0),\ (\ell(y,t);\ t \geq 0,\ y \in \mathbb{R})\right),\end{aligned} \tag{3}$$

where $\ell(y, t)$ is the local time defined in (1). We define the *Brownian bricklayer* to be the process

$$K(t) := \left(B(t),\ \ell(B(t), t)\right), \qquad t \geq 0. \tag{4}$$

In analogy with our discrete-time description, the first coordinate of this process describes the current location of a bricklayer as they move up and down the line, and the second describes the height of the wall in front of the bricklayer's current location. Extending the analogy, the wall itself at time t is described by the *local time profile* $(\ell(y, t),\ y \in \mathbb{R})$.

Theorem 1. *The process* $\left(K(t),\ t \geq 0\right)$ *is almost surely (a.s.) path continuous, and its path a.s. fills the upper half plane. I.e.* $\{K(t) : t \geq 0\} = \mathbb{R} \times [0, \infty)$ *almost surely.*

To prove this result, we appeal to two well-known properties of Brownian local times.

(1) Trotter's theorem (Mörters and Peres, 2010, Theorem 6.19): the random field $(\ell(y, t);\ y \in \mathbb{R}, t \geq 0)$ admits an a.s. continuous version. I.e. there is an a.s. event on which the limit $\ell(y, t)$ is defined and continuous at all points $(y, t) \in \mathbb{R} \times [0, \infty)$.
(2) It is a.s. the case that, for every $y \in \mathbb{R}$, the process $(\ell(y, t),\ t \geq 0)$ increases without bound. This is a corollary to Ray's Theorem (Mörters and Peres, 2010, Theorem 6.38) by way of the strong Markov property.

Let Ω' denote an a.s. event on which both of the above properties hold.

Proof of Theorem 1. The continuity of $(K(t),\ t \geq 0)$ follows from Trotter's theorem and the continuity of Brownian motion itself. It remains to show that the path is a.s. space-filling. To that end, fix $(y, s) \in \mathbb{R} \times [0, \infty)$. It suffices to show that for every outcome $\omega \in \Omega'$ there is some $T = T(\omega, y, s)$ for which $\left(B(T), \ell(B(T), T)\right) = (y, s)$.