STAPRO: 8300 Model 3G pp. 1–7 (col. fig: NIL)

ARTICLE IN PRESS

Statistics and Probability Letters xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Improved on-line estimation for gamma process

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HIGHLIGHTS

- Two new recursive estimators are developed for homogeneous gamma process.
- We show that the two estimators are weakly consistent.
- We compare the two new estimators with these of Paroissin (2017) by simulations.

ARTICLE INFO

Article history:
Received 24 January 2018
Received in revised form 27 April 2018
Accepted 25 July 2018
Available online xxxx

Keywords:
On-line estimation
Gamma process
Inverse Gaussian process
Linear estimation
Wiener process

ABSTRACT

In this paper, two new estimators based on the spirit of the best linear unbiased estimators are separately developed for homogeneous gamma process. Both estimators can be computed recursively, and have high efficiency. We compare the two new estimators with these of Paroissin (2017) by simulations, and find that ours have smaller biases and mean squared errors.

1. Introduction

On-line inference has become an indispensable technique in prognostic and health management (Si et al., 2011; Guan et al., 2016; Wang et al., 2018; Xu et al., 2018; Zhou and Xu, 2018). Most of the on-line inference methods are developed based on the Wiener process. However, the Wiener process is suitable only for degradation processes that is not monotone. For the monotone degradation process, the gamma process is more effective and has received wide applications (Lawless and Crowder, 2004; Ye et al., 2014; Guida et al., 2015; Le Son et al., 2016). Recently, Paroissin (2017) considered on-line estimation of the following homogeneous gamma degradation process $\{X(t), t \geq 0\}$:

1. $X(0) = 0$.
2. $X(t)$ has independent increments.
3. The increment $X(t + \Delta t) - X(t)$ follows the gamma distribution with shape parameter $a\Delta t$ and rate parameter b.

Consider n identical items whose degradation of a performance characteristic follows the above homogeneous gamma process, and let $0 = t_0 < t_1 < t_2 < \cdots < t_m < \cdots$ be the measurement time epoches identical for all items, and $X_i(t_j)$ be the degradation value of the ith item at time t_j. Let $Y_{i,j} = X_i(t_j) - X_i(t_{j-1})$ and $s_j = t_j - t_{j-1}, j = 1, \ldots, m, i = 1, \ldots, n$.

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https://doi.org/10.1016/j.spl.2018.07.021
0167-7152/

Please cite this article in press as: Xu A., Shen L., Improved on-line estimation for gamma process. Statistics and Probability Letters (2018), https://doi.org/10.1016/j.spl.2018.07.021.

Then $Y_{i,j}$ follows the gamma distribution with shape parameter as_j and rate parameter b. Following the notation in Paroissin (2017), let $\tilde{Y}_{i,j} = Y_{i,j}/s_j$, $\bar{Y}_{\cdot,j} = \frac{1}{n}\sum_{i=1}^{n}\tilde{Y}_{i,j}$ and $\tau_j^2 = \frac{s_j}{n-1}\sum_{i=1}^{n}\left(\tilde{Y}_{i,j} - \bar{Y}_{\cdot,j}\right)^2$. Then $\mathrm{E}(\bar{Y}_{\cdot,j}) = a/b \doteq \mu$ and $\mathrm{E}(\tau_j^2) = a/b^2 \doteq \sigma^2$. Paroissin (2017) proposed two estimators for μ and σ^2, which can be computed recursively. Out of the two, the following was shown to have better performance:

$$\hat{\mu}_m = \frac{1}{t_m}\sum_{j=1}^{m} s_j\bar{Y}_{\cdot,j} = \hat{\mu}_{m-1} + \frac{s_m}{t_m}\left(\bar{Y}_{\cdot,m} - \hat{\mu}_{m-1}\right), \tag{1}$$

$$\hat{\sigma}_{1,m}^2 = \frac{1}{t_m}\sum_{j=1}^{m} s_j\tau_j^2 = \hat{\sigma}_{1,m-1}^2 + \frac{s_m}{t_m}\left(\tau_m^2 - \hat{\sigma}_{1,m-1}^2\right). \tag{2}$$

Based on the above, a and b at the mth stage can be estimated by $\hat{a}_{1,m} = \hat{\mu}_m^2/\hat{\sigma}_{1,m}^2$ and $\hat{b}_{1,m} = \hat{\mu}_m/\hat{\sigma}_{1,m}^2$, respectively.

Notice that $t_m = \sum_{j=1}^{m}s_j$, we have $\mathrm{E}(\hat{\mu}_m) = \frac{1}{t_m}\sum_{j=1}^{m}s_j\mathrm{E}(\bar{Y}_{\cdot,j}) = \mu$ and $\mathrm{E}(\hat{\sigma}_{1,m}^2) = \frac{1}{t_m}\sum_{j=1}^{m}s_j\mathrm{E}(\tau_j^2) = \sigma^2$. Besides, $\hat{\mu}_m$ and $\hat{\sigma}_{1,m}^2$ are linear combination of $\{\bar{Y}_{\cdot,j}, j = 1,\ldots,m\}$ and $\{\hat{\sigma}_{1,m}^2, j = 1,\ldots,m\}$, respectively. Thus, both $\hat{\mu}_m$ and $\hat{\sigma}_{1,m}^2$ are linear unbiased estimators. Linear unbiased estimator has been widely used in statistics. The merit of linear unbiased estimator is that it can be constructed easily and be derived recursively. For example, least squares estimator for the linear regression model is a linear combination of the observations, and is also the best linear unbiased estimator (BLUE) of the model parameters. The basic idea to select the BLUE is to find the estimator with the smallest variance in a set of linear unbiased estimators. In view of the fact that Paroissin (2017)'s estimators are linear unbiased estimators, the objective of this paper is to improve these estimators by finding the BLUEs of μ and σ^2.

The paper is organized as follows. In Section 2, we propose two new estimators, and the consistency of the estimators is shown. In Section 3, a simulation study is performed to compare the new estimators with these from Paroissin (2017). A data set is analyzed for illustration in Section 4. Finally, we give a conclusion of this paper.

2. New estimators

As mentioned before, both estimators proposed by Paroissin (2017) are linear unbiased estimators. This observation motivates us to find more efficient estimators in the set of linear unbiased estimators of μ and σ^2. For the parameter μ, we consider the set of linear unbiased estimators $U_\mu = \left\{\sum_{j=1}^{m}\alpha_j\bar{Y}_{\cdot,j}, \sum_{j=1}^{m}\alpha_j = 1\right\}$. Then, for any $\hat{\mu} \in U_\mu$,

$$\mathrm{Var}(\hat{\mu}) = \sum_{j=1}^{m}\alpha_j^2\mathrm{Var}\left(\bar{Y}_{\cdot,j}\right) = \sum_{j=1}^{m}\alpha_j^2\sigma^2/(ns_j).$$

Taking the first derivatives of $\mathrm{Var}(\hat{\mu})$ with respect to α_j, and letting them equal zero, we have

$$\alpha_j/(ns_j) - (1 - \alpha_1 - \cdots - \alpha_{m-1})/(ns_m) = 0, \;\; j = 1, 2, \ldots, m-1,$$

which leads to $\alpha_j = s_j/(s_1 + s_2 + \cdots + s_m) = s_j/t_m$. Thus, $\hat{\mu}_m$ in (1) is the best linear unbiased estimator in the set U_μ. Similarly, for the parameter σ^2, we consider the set of linear unbiased estimators $U_{\sigma^2} = \left\{\sum_{j=1}^{m}\beta_j\tau_j^2, \sum_{j=1}^{m}\beta_j = 1\right\}$. From Paroissin (2017), we have

$$\mathrm{Var}(\tau_j^2) = \frac{2a^2}{(n-1)b^4} + \frac{6a}{nb^4s_j} = V_1(1 + V_0/s_j),$$

where $V_1 = \frac{2a^2}{(n-1)b^4}$ and $V_0 = \frac{3(n-1)}{na}$. For any $\hat{\sigma}^2 \in U_{\sigma^2}$,

$$\mathrm{Var}(\hat{\sigma}^2) = \sum_{j=1}^{m}\beta_j^2\mathrm{Var}\left(\tau_j^2\right) = \sum_{j=1}^{m}\beta_j^2V_1(1 + V_0/s_j).$$

Taking the first derivatives of $\mathrm{Var}(\hat{\sigma}^2)$ with respect to β_j, and letting them equal zero, we have

$$\beta_j(1 + V_0/s_j) - (1 - \beta_1 - \cdots - \beta_{m-1})(1 + V_0/s_m) = 0, \;\; j = 1, 2, \ldots, m-1,$$

which follows that

$$\beta_j = \frac{(1 + V_0/s_j)^{-1}}{\sum_{j=1}^{m}(1 + V_0/s_j)^{-1}}. \tag{3}$$

Thus, if the shape parameter a is known, then the best linear unbiased estimator in the set U_{σ^2} is

$$\hat{\sigma}_m^2 = \frac{\sum_{j=1}^{m}(1 + V_0/s_j)^{-1}\tau_j^2}{\sum_{j=1}^{m}(1 + V_0/s_j)^{-1}} = \hat{\sigma}_{m-1}^2 + \frac{(1 + V_0/s_m)^{-1}(\tau_m^2 - \hat{\sigma}_{m-1}^2)}{\sum_{j=1}^{m}(1 + V_0/s_j)^{-1}}.$$

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